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THE FUNCTIONAL EQUATION
$af(ax) + bf(bx + a) = bf(bx) + af(ax + b)$
EXTENSIONS AND ALMOST PERIODIC SOLUTIONS

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Composition operators play a central role in the study of many naturally defined classes of linear transformations. They are often extremal elements in a geometric or analytic sense. A simple class of such operators acting on $L^1(I)$, where $I = [0, 1]$ with Lebesgue measure, consists of those determined by composition with measure preserving transformations $\phi: I \longrightarrow I$. The operator T_ϕ is then defined by $T_\phi f = f \circ \phi$. If we consider the convex class $\mathcal{D}$ of all operators defined on $L^1(I)$ which satisfy (i) $T1 = 1$, (ii) $\int_I Tf = \int_I f$, (iii) $Tf \geqslant 0$ whenever $f \geqslant 0$, then the composition operators are among the extreme points of $\mathcal{D}$ [1]. Closely associated with such a convex class of operators are the orbits Ω of various elements of $L^1(I)$: $\Omega(f) = \{Tf: T \in \mathcal{D}, f \text{ fixed in } L^1(I)\}$. The extreme points of each orbit are known [2], [3] and the question arises (applicable to other classes of operators as well):

Does there exist an extreme operator in $\mathcal{D}$ which preserves no extreme points of any orbit? That is, does there exist an extreme element of $\mathcal{D}$ such that whenever g is an extreme point of $\Omega(f)$, then Tg is not extreme? This should hold true for all nonconstant $f \in L^1(I)$ (see [5]).

It is enough to show that an extreme operator can be found which preserves no characteristic functions except for the constant functions 1 and 0 [4, Lemma 4]. To this end we consider two (noninvertible) measure preserving transformations:

$$\phi(x) = \begin{cases} x/a, & 0 \leqslant x \leqslant a, \\ (x-a)/b, & a \leqslant x \leqslant 1, \end{cases} \qquad \psi(x) = \begin{cases} x/b, & 0 \leqslant x \leqslant b, \\ (x-b)/a, & b \leqslant x \leqslant 1, \end{cases}$$

where $a + b = 1$. In order to define a mapping on $L^1(I)$ which fails to preserve characteristic functions, choose a function γ, $0 < \gamma < 1$, and set

$$T = \gamma T_\phi + (1 - \gamma) T_\psi. \tag{1}$$

We will leave the class $\mathcal{D}$ unless γ is also chosen so that (ii) is satisfied. This

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will be achieved whenever γ is a solution to the functional equation

$$\text{(E)} \qquad af(ax) + bf(bx + a) = bf(bx) + af(ax + b), \qquad a + b = 1.$$

The operator T will be extreme in $\mathcal{D}$ if, in addition, we know that γ is an extreme point in the set of solutions of (E) subject to $0 \leqslant f \leqslant 1$. Thus one is led to the problem of finding all bounded solutions of (E), determining the extreme points in $\{f\colon 0 \leqslant f \leqslant 1\}$, and then checking whether or not an extreme point exists which is not equal to 1 (or 0) on a set of positive measure. The last property is necessary in order that the operator T in (1) not preserve any characteristic functions. The equation (E), incidentally, connects the adjoints of T_ϕ and T_ψ and is the same as asking for solutions to $T_\phi^* f = T_\psi^* f$.

Current results. Regarding (E), we note that the case $a = b$ is trivial, and one sees immediately that the equation is satisfied by any linear function.

Moreover, if a, hence b, is rational there are periodic solutions.

The first significant contribution to this question was made by J. H. Dhombres, the proof of which will appear elsewhere.

THEOREM (DHOMBRES). *If $a = 1/n$ $(n > 2)$ and f is a Riemann integrable solution of* (E), *then $f(x) = \alpha x + \beta + p(x)$ where p is periodic of period* $1/n$.

This discovery tended to support the conjecture that in the case where a is irrational, only linear solutions exist. The conjecture does not even remotely reflect the true state of affairs.

Integrating the functional equation, one obtains

$$\text{(E*)} \qquad F(ax) + F(bx + a) = F(bx) + F(ax + b), \qquad F = \int_0^x f,$$

where a constant has been added to f to yield $F(a) = F(b)$ (the constant is $\int_a^b f$).

We can now list some positive assertions regarding (E*). In what follows, the relationship $0 < a < b < 1$ will hold.

PROPOSITION 1. *If F is a continuous solution to* (E*) *defined on I there exist extensions of F to* **R** *which remain solutions to* (E*).

In fact there is a convex set of extensions which, on the interval $[1 + a^2/b,\ 1 + a]$ can be any function whatsoever.

Despite this generality one still has

PROPOSITION 2. *If F is a continuous solution to* (E*) *defined on I, then F has a unique continuous extension to* **R** *which satisfies* (E*).

This does not address the question of existence of solutions nor of their differentiability (in order to return to (E)). An abundance of analytic solutions is guaranteed by the next result.

PROPOSITION 3. *The almost periodic functions*

$$\phi_m(x) = (1 - \exp(2\pi im/b)) \exp(2\pi imx/a) - (1 - \exp(2\pi im/a)) \exp(2\pi imx/b), \qquad m = 0, \pm 1, \pm 2, \ldots,$$

are solutions to (E*).

It is clear that certain periodic functions are obtained when a is rational. Polynomial solutions, on the other hand, are scarce. In fact, it is not difficult to show that the only polynomial solution to (E*) must have the form $q(x) = \alpha(x^2 - x) + \beta$. These quadratics play an interesting, albeit mysterious, role in the overall picture.

PROPOSITION 4. *The only universal solutions of* (E*), *i.e. solutions for all admissible values of a and b, of class $C^1(I)$, are the above polynomials q.*

If we are given a quadratic solution to (E*) restricted to I, then we know that it has a unique continuous extension to $\mathbf{R}$—the same quadratic. However, quadratics are not almost periodic. The next assertion indicates that we still do not have the final chapter regarding (E*).

THEOREM. *If f is an almost-periodic solution to* (E*), *then $f = \Sigma_m a_m \phi_m$ where ϕ_m is defined in Proposition* 3.

Perhaps the best conjecture one can suggest is that every reasonable solution is a quadratic q plus an almost periodic solution.

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