

GENERATORS OF THE UNITARY \mathbf{Z}/p BORDISM RING

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Communicated by P. T. Church, December 29, 1975

I. **Serendipity.** Let p be a prime number, let \mathbf{Z}/p be the cyclic group of order p and let $\mathfrak{A}_*^{\mathbf{Z}/p}$ be the unitary \mathbf{Z}/p bordism ring.

THEOREM 1. $\mathfrak{A}_*^{\mathbf{Z}/p}$ is multiplicatively generated, over \mathfrak{A}_* , by the following:

- $\{\Gamma^m(\text{pt}), m \geq 0\}$,
- $\mathbf{U}\{\mathbf{Z}/p\}$,
- $\mathbf{U}\{\Gamma^m(\mathbf{C}P_j^1), m \geq 0, (p+1)/2 \leq j < p-1\}$,
- $\mathbf{U}\{S_j, 1 \leq j \leq (p-1)/2\}$,
- $\mathbf{U}\{\Gamma^m(C_j), m \geq 0, 1 < j \leq (p+1)/2\}$,
- $\mathbf{U}\{\Gamma^m(\mathbf{C}P_j^n), m \geq 0, n \geq 2, 1 \leq j \leq p-1\}$.

Furthermore, this set is irredundant.

The notation is explained by the following.

- (a) pt the point, with obvious \mathbf{Z}/p action.
- (b) \mathbf{Z}/p , p points with obvious \mathbf{Z}/p action.
- (c) $\mathbf{C}P_j^1$, $((p+1)/2 \leq j < p-1)$, the complex projective, line $\mathbf{C}P^1$ with \mathbf{Z}/p action given by $[z_0; z_1] \mapsto [z_0; \xi^j z_1]$ where $\xi = \exp(2\pi i/p)$.
- (d) S_j , $(1 \leq j \leq (p-1)/2)$, the Riemann surface of genus $(q-1)(p-1)/2$ associated to the complex function $u = (z^p - 1)^{1/q}$ where q satisfies $qj = -1 \pmod p$, $0 < q < p$. The action of \mathbf{Z}/p on S_j is induced by $z \mapsto \xi z$.
- (e) C_j , $(1 < j \leq (p+1)/2)$, the complex projective plane $\mathbf{C}P^2$ with \mathbf{Z}/p action given by $[z_0; z_1; z_2] \mapsto [z_0; \xi z_1; \xi^j z_2]$.
- (f) $\mathbf{C}P_j^n$, $(n \geq 2, 1 \leq j \leq p-1)$, the complex projective space $\mathbf{C}P^n$ with \mathbf{Z}/p action given by $[z_0; z_1; \dots; z_{n-1}; z_n] \mapsto [z_0; z_1; \dots; z_{n-1}; \xi^j z_n]$.
- (g) Let M be a unitary \mathbf{Z}/p manifold for which the \mathbf{Z}/p action extends to a unitary S^1 action. For example, the unitary \mathbf{Z}/p manifolds in (a), (c), (e) and (f) satisfy this property.

The circle S^1 acts freely on the product $M \times S^3$ by $(m, z_1, z_2) \mapsto (tm, tz_1, tz_2)$, $t \in S^1$, where $S^3 = \{(z_1, z_2) \in \mathbf{C}^2; |z_1|^2 + |z_2|^2 = 1\}$. Let $\Gamma(M)$ denote the quotient $(M \times S^3)/S^1$, with \mathbf{Z}/p action given by $[m, z_1, z_2] \mapsto [\xi m, \xi z_1, z_2]$. Of course, this action extends to an S^1 action and we can define,

AMS (MOS) subject classifications (1970). Primary 57D85, 57E25; Secondary 57D20.
Key words and phrases. Equivariant bordism, \mathbf{Z}/p manifolds, characteristic numbers.

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inductively, $\Gamma^n(M)$ by $\Gamma^n(M) = \Gamma(\Gamma^{n-1}(M))$. Also we set $\Gamma^0(M)$ to be M .

II. Applications. Note that most of the generators of $\mathfrak{A}_*^{\mathbf{Z}/p}$ are in fact S^1 manifolds and so we can use known results concerning S^1 manifolds to obtain corresponding results for \mathbf{Z}/p manifolds. The following mod p characteristic numbers formula is the analogue of the formula for S^1 manifolds given in §8 of [1].

THEOREM 2. *Suppose M is a unitary \mathbf{Z}/p manifold of dimension $2n$ and suppose that f is any symmetric homogeneous polynomial in n variables of degree less than or equal to n , then*

$$\sum \{f(y_1, y_2, \dots, y_d, t_1 + z_1, t_2 + z_2, \dots, t_{n-d} + z_{n-d}) \prod (t_j + z_j)^{-1}\} [F] \\ = f(w_1, w_2, \dots, w_n) [M] \text{ mod } p$$

where the sum is taken over the components F of the fixed point set. The t_i are the “rotation numbers”, and elementary symmetric functions of y_i, z_i and w_i respectively, give the chern classes of F , normal bundle of F and M respectively ($\dim F = 2d$).

Such a formula was known [2] but in the case that each component of the fixed point set has dimension less than $2(p - 1)$, we require no such restrictions.

Let s_n be the polynomial defined by $s_n(x_1, x_2, \dots, x_n) = x_1^n + x_2^n + \dots + x_n^n$; then we can say more.

THEOREM 3. *Suppose M is a \mathbf{Z}/p manifold of dimension $2n$, then*

$$\sum \{s_n(y_1, y_2, \dots, y_d, t_1 + z_1, t_2 + z_2, \dots, t_{n-d} + z_{n-d}) \prod (t_j + z_j)^{-1}\} [F] \\ = \begin{cases} s_n [M] \text{ mod } p^2 & \text{if } n = p^k - 1 \text{ for some } k, \\ s_n [M] \text{ mod } p & \text{otherwise.} \end{cases}$$

Using Theorems 1 and 3 we obtain

THEOREM 4. *Let M be a \mathbf{Z}/p manifold of dimension $2n$ such that*

- (i) $n \equiv 0 \pmod{p - 1}$,
 - (ii) either $n \equiv -1 \pmod{p}$ or else each component of the fixed point set has trivial normal bundle in M ,
 - (iii) no component of the fixed point set is of dimension $2n$, and
 - (iv) $n \neq p^k - 1$ for any $k \geq 0$;
- then M is decomposable mod p , i.e. decomposable as an element of $\mathfrak{A}_*/p\mathfrak{A}_*$.

As an example, \mathbf{Z}/p manifolds of dimension $2n = 2(kp + 1)(p - 1)$, where $k > 0$ and $k \not\equiv 1 \pmod{p}$, which have no component of the fixed point set of dimension $2n$ are decomposable mod p .

Finally, we mention a result whose proof uses a free \mathfrak{A}_* basis for $\mathfrak{A}_*^{\mathbb{Z}/p}$.

THEOREM 5. *Let M be a \mathbb{Z}/p manifold such that*

- (i) *no component of M has a trivial \mathbb{Z}/p action,*
- (ii) *each component of the fixed point set has trivial normal bundle and*
- (iii) *M has no isolated fixed points;*

then M is equivariantly decomposable modulo free \mathbb{Z}/p manifolds, i.e. decomposable in $\mathfrak{A}_^{\mathbb{Z}/p}/p\mathfrak{A}_*$.*

The fact that such manifolds are decomposable in $\mathfrak{A}_*/p\mathfrak{A}_*$ follows quite easily from Theorem 3.

All the above results have obvious analogues in the oriented case so long as p is an odd prime.

Details of proofs will appear elsewhere.

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