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ON STOPPING TIME DIRECTED CONVERGENCE

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The main purpose of this note is to introduce the notion of $\bar{S}$-martingales, a certain modification of that of asymptotic martingales, the main justification of which is III.

1. **S-convergence.** Let (Ω, F, P) be a probability; (F_n) $(n = 1, 2, \ldots)$, a nondecreasing sequence of measurable σ-fields and (X_n) an adapted sequence of extended real-valued r.v. (random variables). (If the F_n are not mentioned explicitly then any F_n with the above properties will do; in particular, we may take F_n to be the σ-field generated by $X_1, \ldots, X_n$.) Let $T = \{t\}$ be the family of *bounded stopping times*; i.e. the family of positive, bounded, integer-valued r.v. t with $t^{-1}(n) \in F_n$ for all n. T is a directed set filtering to the right under the relation $t_1 \leqslant t_2$, i.e. $t_1(\omega) \leqslant t_2(\omega)$ a.s. (almost surely). The r.v. X_t for $t \in T$, is defined by $X_t(\omega) = X_{t(\omega)}(\omega)$.

DEFINITION. Let ϕ map X_t $(t \in T)$ into a topological space M. Then $(\phi(S_n))$ is said to be *S-convergent*—or stopping time directed convergent—(to Y) if the directed set $\phi(X_t)$ is convergent in the topology of M (to Y).

S-convergence implies ordinary convergence, but not vice-versa.

EXAMPLES. (1) ϕ the identity mapping, M the space of all extended real valued r.v. topologized by convergence in probability (for extended real valued r.v. this is interpreted as applied to the r.v. obtained through the mapping $x \to x/(1 + |x|)$). We then speak of *S-convergence in probability.* In sharp distinction from the situation in ordinary convergence, we have

I. *S-convergence in probability is equivalent to a.s. convergence.*

The proof is immediate since there exist $t_1 < \cdots < t_n < \cdots \to \infty$ with $X_{t_n} \xrightarrow{\text{a.s}} \limsup X_{t_n} = \limsup X_n$.

(2) (X_n) is said to be an *S-martingale* if the expectations (finite or not) EX_t are defined for all $t \in T$ and (EX_n) is S-convergent (to a finite or infinite number). If the limit is a finite number then (X_n) is called an *asymptotic martingale.*

The argument proving I yields

II. *A uniformly bounded sequence of r.v. (X_n) is a.s. convergent iff it is an asymptotic martingale.*

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This result is due to P. A. Meyer [5] where, apparently, stopping time directed convergence was first considered in this context. Asymptotic martingales were introduced by D. G. Austin, G. A. Edgar and A. Ionescu Tulcea [1] and their properties further studied and importance underlined by R. V. Chacon and L. Sucheston [3], G. A. Edgar and L. Sucheston [4] and A. Bellow [2].

2. **$\bar{S}$-convergence.** Let C^H $(0 \leqslant H < \infty)$ be defined by: $C^H(x) = x$ if $|x| \leqslant H$ and $C^H(x) = \pm H$ otherwise, according to the sign of x.

DEFINITION. Let φ map uniformly bounded r.v. into the topological space M. Then $(\phi(X_n))$ is said to be *$\bar{S}$-convergent* if $\phi(C^H X_n)$ is S-convergent for every H.

In particular, (X_n) is said to be an *$\bar{S}$-martingale* if $(C^H X_n)$ is an asymptotic martingale for every H. *$\bar{S}$-convergence* in probability is defined as S-convergence in probability of $(C^H X_n)$ for every finite positive H.

From II we immediately have

III. *A sequence of extended real-valued r.v. (X_n) is a.s. convergent iff it is an $\bar{S}$-martingale.*

It follows (and can also be shown directly) that in defining $\bar{S}$-martingales it suffices to consider any unbounded set of H's. On the other hand, one can equivalently define $\bar{S}$-martingales by the requirement that $(f(X_n))$ is an asymptotic martingale for every bounded continuous function f from the extended real line into the real line.

We remark that

IV. *The limit function in* III *is a.s. finite iff*

$$\lim_{H\to\infty} \limsup_{n\to\infty} P[|X_n| > H] = 0$$

From III we have immediately

V. (a) *If (X_n) and (Y_n) are $\bar{S}$-martingales, then* $(\min(X_n, Y_n))$ *and* $(\max(X_n, Y_n))$ *are also $\bar{S}$-martingales.*

(b) *A function f transforms every $\bar{S}$-martingale (X_n) into an $\bar{S}$-martingale $(f(X_n))$ iff it is continuous* (on the extended real line).

(c) *If P' is a probability measure defined on $\mathcal{F}$ then every $\bar{S}$-martingale on $(\Omega, \mathcal{F}, P)$ is an $\bar{S}$-martingale on $(\Omega, \mathcal{F}, P')$ iff P' is absolutely continuous relative to P.*

3. **Further results.** In [1] (see also [4], [2]) it was proved that an L_1-bounded asymptotic martingale is a.s. convergent. This implies

VI. *An L_1-bounded asymptotic martingale is an $\bar{S}$-martingale.*

This also follows directly from VII, which extends a result of [1].

VII. If (X_n) *is an asymptotic martinglae then* (X_n^+) and (X_n^-) *are S-martingales.*

To see this note that if $t' \geqslant t$ and we put $t'' = t'$ on $[X_t \geqslant 0]$ and $t'' = t$ otherwise, then $t'' \geqslant t$. Hence, if t is such that $EX_{t'} - EX_t \geqslant -\epsilon$ for all $t' \geqslant$

t, it follows that $EX_{t'}^+ - EX_t^+ \geqslant EX_{t''}^+ - EX_t^+ \geqslant EX_{t''} - EX_t \geqslant -\epsilon$.

We remark that the conclusion of VII holds even for S-martingales which are not asymptotic martingales, provided the approach of EX_t to infinity is "semimonotone". In particular it holds for super- and sub-martingales (even not L_1-bounded).

4. **Generalizations**. There are several possible ways of generalizing the above results. Thus (1) more general directed sets than sequences can be considered (see [4]). (2) Vector valued (and other) r.v. may be considered (see [3], [4], [2]). (3) Probability spaces can be replaced by σ-finite measure spaces. (4) Approximate (see [4]) S- and $\bar{S}$-martingales can be studied.

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