ON STOPPING TIME DIRECTED CONVERGENCE

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The main purpose of this note is to introduce the notion of $S$-martingales, a certain modification of that of asymptotic martingales, the main justification of which is III.

1. $S$-convergence. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability; $(\mathcal{F}_n) \ (n = 1, 2, \ldots)$, a nondecreasing sequence of measurable $\sigma$-fields and $(X_n)$ an adapted sequence of extended real-valued r.v. (random variables). (If the $\mathcal{F}_n$ are not mentioned explicitly then any $\mathcal{F}_n$ with the above properties will do; in particular, we may take $\mathcal{F}_n$ to be the $\sigma$-field generated by $X_1, \ldots, X_n$.) Let $T = \{t\}$ be the family of bounded stopping times; i.e. the family of positive, bounded, integer-valued r.v. $t$ with $t^{-1}(n) \in \mathcal{F}_n$ for all $n$. $T$ is a directed set filtering to the right under the relation $t_1 < t_2$, i.e. $t_1(\omega) \leq t_2(\omega)$ a.s. (almost surely). The r.v. $X_t$ for $t \in T$, is defined by $X_t(\omega) = X_{t(\omega)}(\omega)$.

**Definition.** Let $\phi$ map $X_t \ (t \in T)$ into a topological space $M$. Then $(\phi(S_n))$ is said to be $S$-convergent—or stopping time directed convergent—(to $Y$) if the directed set $\phi(X_t)$ is convergent in the topology of $M$ (to $Y$).

$S$-convergence implies ordinary convergence, but not vice-versa.

**Examples.** (1) $\phi$ the identity mapping, $M$ the space of all extended real valued r.v. topologized by convergence in probability (for extended real valued r.v. this is interpreted as applied to the r.v. obtained through the mapping $x \mapsto x/(1 + |x|)$). We then speak of $S$-convergence in probability. In sharp distinction from the situation in ordinary convergence, we have

I. **$S$-convergence in probability is equivalent to a.s. convergence.**

The proof is immediate since there exist $t_1 < \cdots < t_n < \cdots \to \infty$ with $X_{t_n} \overset{a.s.}{\to} \limsup X_{t_n} = \limsup X_n$.

(2) $(X_n)$ is said to be an $S$-martingale if the expectations (finite or not) $EX_t$ are defined for all $t \in T$ and $(EX_n)$ is $S$-convergent (to a finite or infinite number). If the limit is a finite number then $(X_n)$ is called an asymptotic martingale.

The argument proving I yields

II. **A uniformly bounded sequence of r.v. $(X_n)$ is a.s. convergent iff it is an asymptotic martingale.**

This result is due to P. A. Meyer [5] where, apparently, stopping time directed convergence was first considered in this context. Asymptotic martingales were introduced by D. G. Austin, G. A. Edgar and A. Ionescu Tulcea [1] and their properties further studied and importance underlined by R. V. Chacon and L. Sucheston [3], G. A. Edgar and L. Sucheston [4] and A. Bellow [2].

2. \( \bar{S} \)-convergence. Let \( C^H \) \((0 \leq H < \infty)\) be defined by: \( C^H(x) = x \) if \(|x| \leq H \) and \( C^H(x) = \pm H \) otherwise, according to the sign of \( x \).

**Definition.** Let \( \varphi \) map uniformly bounded r.v. into the topological space \( M \). Then \( (\varphi(X_n)) \) is said to be \( \bar{S} \)-convergent if \( \varphi(C^HX_n) \) is \( S \)-convergent for every \( H \).

In particular, \( (X_n) \) is said to be an \( \bar{S} \)-martingale if \( (C^HX_n) \) is an asymptotic martingale for every \( H \). \( \bar{S} \)-convergence in probability is defined as \( S \)-convergence in probability of \( (C^HX_n) \) for every finite positive \( H \).

From II we immediately have

**III.** A sequence of extended real-valued r.v. \( (X_n) \) is a.s. convergent iff it is an \( \bar{S} \)-martingale.

It follows (and can also be shown directly) that in defining \( \bar{S} \)-martingales it suffices to consider any unbounded set of \( H \)'s. On the other hand, one can equivalently define \( \bar{S} \)-martingales by the requirement that \( (f(X_n)) \) is an asymptotic martingale for every bounded continuous function \( f \) from the extended real line into the real line.

We remark that

**IV.** The limit function in III is a.s. finite iff

\[
\lim_{H \to \infty} \limsup_{n \to \infty} P[|X_n| > H] = 0
\]

From III we have immediately

**V.** (a) If \( (X_n) \) and \( (Y_n) \) are \( \bar{S} \)-martingales, then \( (\min(X_n, Y_n)) \) and \( (\max(X_n, Y_n)) \) are also \( \bar{S} \)-martingales.

(b) A function \( f \) transforms every \( \bar{S} \)-martingale \( (X_n) \) into an \( \bar{S} \)-martingale \( (f(X_n)) \) iff it is continuous (on the extended real line).

(c) If \( P' \) is a probability measure defined on \( \mathcal{F} \) then every \( \bar{S} \)-martingale on \( (\Omega, \mathcal{F}, P) \) is an \( \bar{S} \)-martingale on \( (\Omega, \mathcal{F}, P') \) iff \( P' \) is absolutely continuous relative to \( P \).

3. **Further results.** In [1] (see also [4], [2]) it was proved that an \( L_1 \)-bounded asymptotic martingale is a.s. convergent. This implies

**VI.** An \( L_1 \)-bounded asymptotic martingale is an \( \bar{S} \)-martingale.

This also follows directly from VII, which extends a result of [1].

**VII.** If \( (X_n) \) is an asymptotic martingale then \( (X_n^+) \) and \( (X_n^-) \) are \( S \)-martingales.

To see this note that if \( t' \geq t \) and we put \( t'' = t' \) on \([X_t' \geq 0]\) and \( t'' = t \) otherwise, then \( t'' \geq t \). Hence, if \( t \) is such that \( EX_{t'} - EX_t \geq - \varepsilon \) for all \( t' \geq
t, it follows that $EX_t^+ - EX_t^- \geq EX_t^+ - EX_{t'}^+ \geq EX_{t'}^- - EX_t^- > - \varepsilon$.

We remark that the conclusion of VII holds even for $S$-martingales which are not asymptotic martingales, provided the approach of $EX_t$ to infinity is "semimonotone". In particular it holds for super- and sub-martingales (even not $L_1$-bounded).

4. Generalizations. There are several possible ways of generalizing the above results. Thus (1) more general directed sets than sequences can be considered (see [4]). (2) Vector valued (and other) r.v. may be considered (see [3], [4], [2]). (3) Probability spaces can be replaced by $\sigma$-finite measure spaces. (4) Approximate (see [4]) $S$- and $\tilde{S}$-martingales can be studied.

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References


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