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EXISTENCE AND COMPARISON OF EIGENVALUES OF nTH ORDER LINEAR DIFFERENTIAL EQUATIONS

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Consider the differential equations

$$[a(x)u^{(k)}(x)]^{(n-k)} - (-1)^{n-k}\lambda \sum_{i=0}^{k-1} p_i(x)u^{(i)}(x) = 0 \tag{1.1}$$

and

$$[A(x)v^{(k)}(x)]^{(n-k)} - (-1)^{n-k}\Lambda \sum_{i=0}^{k-1} q_i(x)v^{(i)}(x) = 0, \tag{1.2}$$

subject to the boundary conditions

$$\begin{aligned} u(\alpha) &= u'(\alpha) = \cdots = u^{(k-1)}(\alpha) \\ &= u_1(\beta) = u_1'(\beta) = \cdots = u_1^{(n-k-1)}(\beta) = 0, \end{aligned} \tag{1.1a}$$

$$\begin{aligned} v(\alpha) &= v'(\alpha) = \cdots = v^{(k-1)}(\alpha) \\ &= v_1(\beta) = v_1'(\beta) = \cdots = v_1^{(n-k-1)}(\beta) = 0, \end{aligned} \tag{1.2a}$$

or

$$\begin{aligned} u(\alpha) &= u'(\alpha) = \cdots = u^{(k-1)}(\alpha) \\ &= u(\beta) = u'(\beta) = \cdots = u^{(n-k-1)}(\beta) = 0, \end{aligned} \tag{1.1b}$$

$$\begin{aligned} v(\alpha) &= v'(\alpha) = \cdots = v^{(k-1)}(\alpha) \\ &= v(\beta) = v'(\beta) = \cdots = v^{(n-k-1)}(\beta) = 0, \end{aligned} \tag{1.2b}$$

where $u_1(x) \equiv a(x)u^{(k)}(x)$ and $v_1(x) \equiv A(x)v^{(k)}(x)$. We assume that the functions $a(x)$, $p_0(x)$, and $q_0(x)$ are positive on $[\alpha, \beta]$. Equations (1.1)–(1.2) subject to the boundary conditions (1.1a)–(1.2a) [(1.2b)–(1.2b), respectively] will be termed the $(k, n-k)$-focal point eigenvalue problem [the $(k, n-k)$-conjugate point eigenvalue problem, respectively].

We establish the existence of a smallest positive eigenvalue for both of these eigenvalue problems and prove comparison theorems relating the eigenvalues. Our

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results generalize the classical selfadjoint eigenvalue comparison theorems in two distinct ways. First, we allow the principal part of the differential equation to be of arbitrary order; thereby, allowing the problem to be nonselfadjoint. Second, even in the case where the principal part of the differential equation is formally selfadjoint, that is, when $n = 2k$, the eigenvalue problem does not reduce to a selfadjoint eigenvalue problem unless $p_i(x) \equiv 0$ for $i = 1, 2, \ldots, k-1$. Only in this case does the Courant Minimum Principle apply yielding that the last positive eigenvalues of (1.1) and (1.2) satisfy $\Lambda_0 \leqslant \lambda_0$, if $0 < A(x) \leqslant a(x)$ and $p_0(x) \leqslant q_0(x)$ on $[\alpha, \beta]$. The following are an example of the type of results we have obtained for equations (1.1) and (1.2).

THEOREM 1.1. *Assume that* $0 \leqslant \int_x^\beta p_i(s)\, ds$ *on* $[\alpha, \beta]$ *for* $i = 0, 1, \ldots, k-1$; $[0 = p_i(x)$ *on* $[\alpha, \beta]$ *for* $i = 1, \ldots, k-1$, *respectively*$]$; *then the* $(k, n-k)$*-focal point eigenvalue problem* [*the* $(k, n-k)$*-conjugate point eigenvalue problem, respectively*] *has at least one real eigenvalue which is positive and smaller than the absolute value of any other eigenvalue. The eigenfunction associated with this eigenvalue is positive on* (α, β).

THEOREM 1.2. *If*

(i) $$0 \leqslant p_i(x) \quad \text{on } [\alpha, \beta] \text{ for } i = 0, \ldots, k-1,$$

(ii) $$\int_x^\beta p_i(s)\, ds \leqslant \int_x^\beta q_i(s)\, ds \quad \text{on } [\alpha, \beta] \text{ for } i = 0, 1, \ldots, k-1,$$

(iii) $$\int_\alpha^x a(s)\, ds \leqslant \int_\alpha^x A(s)\, ds \quad \text{on } [\alpha, \beta],$$

then the smallest positive eigenvalues λ_0 *and* Λ_0 *of the* $(k, n-k)$*-focal point eigenvalue problems* (1.1) *and* (1.2), *respectively, satisfy,* $\Lambda_0 \leqslant \lambda_0$ *with equality if and only if* $a(x) \equiv A(x)$ *and* $p_i(x) \equiv q_i(x)$ *on* $[\alpha, \beta]$ *for* $i = 0, 1, \ldots, k-1$.

Eigenvalue comparison theorems of the "integral type", such as appear in Theorem 1.2, were first established by Z. Nehari [1] for second order differential equations, and later extended to selfadjoint equations of order $2n$ by C. C. Travis [2].

THEOREM 1.3. *If*

(i) $$0 < p_0(x) \leqslant q_0(x),$$

(ii) $$0 = p_i(x) = q_i(x) \quad \text{on } [\alpha, \beta] \text{ for } i = 1, 2, \ldots, k-1,$$

(iii) $$0 < A(x) \leqslant a(x) \quad \text{on } [\alpha, \beta],$$

then the smallest positive eigenvalues λ_0 *and* Λ_0 *of the* $(k, n-k)$*-conjugate point eigenvalue problems* (1.1) *and* (1.2), *respectively, satisfy* $\Lambda_0 \leqslant \lambda_0$ *with equality if and only if* $a(x) \equiv A(x)$ *and* $p_0(x) \equiv q_0(x)$ *on* $[\alpha, \beta]$.

Proofs and applications of the above results will appear elsewhere.

REFERENCES

1. Z. Nehari, *Oscillation criteria for second order linear differential equations*, Trans. Amer. Math. Soc. 85 (1957), 428–445. MR 19, 415.

2. C. C. Travis, *Comparison of eigenvalues for linear differential equations of order* $2n$, Trans. Amer. Math. Soc. 177 (1973), 363–374. MR 47 #5356.

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