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Scattering theory for the d'Alembert equation in exterior domains, by Calvin H. Wilcox, Lecture Notes in Mathematics, No. 442, Springer-Verlag, Berlin, Heidelberg, New York, 1975, 184 pp., \$8.60.

The subject of this book is a mathematical model for the propagation of sound around obstacles. The basic problem is to describe the behavior of sound waves which impinge on an infinitely hard object occupying a compact region $\Gamma \subset \mathbf{R}^3$ (the analysis is carried out for $\Gamma \subset \mathbf{R}^n$). Roughly, one has an incoming wave u_- which is unaffected by the obstacle. The sound then reaches Γ where it is reflected, diffracted, and is generally subject to complicated physical processes. Eventually the intensity of sound near Γ dies out indicating that the sound wave has traveled away from Γ (the model is conservative so the only way for sound to disappear in one place is for it to appear somewhere else). Thus for large time one expects to find a wave u_+ which is unaffected by the obstacle.

The mathematical model is the following. The sound wave is described by a function $u: \mathbf{R} \times (\mathbf{R}^3 \backslash \Gamma) \to \mathbf{R}$ where $\partial u(t, x)/\partial t$ represents the difference between the pressure at place x and time t and the equilibrium pressure. With an appropriate choice of units the equation of motion for u is the wave (or d'Alembert) equation,

$$u_{tt} - \Delta u = 0 \tag{1}$$

where $\Delta = \sum_{i=1}^n (\partial/\partial x_i)^2$. The Neumann boundary condition

$$\mathbf{n} \cdot \operatorname{grad}_x u = 0 \quad \text{on } \mathbf{R} \times \partial\Gamma \tag{2}$$

($\mathbf{n}$=normal to $\partial\Gamma$) describes the interaction of sound with an infinitely hard obstacle. Waves in the absence of obstacles satisfy d'Alembert's equation on the entire space $\mathbf{R} \times \mathbf{R}^n$.

The intuition described above suggests that if u is a solution of (1), (2), then on any bounded set $\beta \subset \mathbf{R}^n$, $u(t)|_\beta \to 0$ as $t \to \infty$ and that there is a solution, u_+, of the wave equation on $\mathbf{R} \times \mathbf{R}^n$ with $u \approx u_+$ for t large. Similar assertions hold for $t \to -\infty$ with an associated free wave u_-. From a practical perspective one often knows the initial free wave u_- which then interacts with Γ then tends to u_+ as $t \to \infty$. The map $u_- \to u_+$ is called the scattering operator.

The first problems of scattering theory are to show that for any solution, u, of (1), (2) there is u_+ with $u \approx u_+$ for $t \gg 1$ (existence of wave operators) and that for any free wave u_- there is a u satisfying (1), (2) with $u \approx u_-$ for $t \ll -1$ (completeness of wave operators). One then proceeds with a more detailed analysis of the relations among $u_\pm$ and u. One of the main goals, which at present has not been reached, is the inverse problem: given the scattering operator find the obstacle, Γ. For example in the echo location of objects one sends out a free wave, u_-, and observes (a part of) the scattered wave u_+ and from the information so obtained reconstruction of Γ is attempted. The rudimentary nervous system of a bat is capable of solving this problem with exceptional skill but man's theoretical and practical attempts fall far short of the bat's achievements.

There are several approaches to the study of scattering and all begin by casting the equation of motion in the form $dU/dt = BU$ where U is a function on $\mathbf{R}$ with values in a suitable Banach space, the vector $U(t)$ represents the state of the system at time t. The operator B is always discontinuous. In the present situation the underlying space is the Hilbert space $H_1(\mathbf{R}^n \setminus \Gamma) \oplus L_2(\mathbf{R}^n \setminus \Gamma)$, $U(t) = (u(t), \partial u(t)/\partial t)$ and $B = \left(\begin{smallmatrix} 0 & 1 \\ \Delta & 0 \end{smallmatrix}\right)$ is selfadjoint on the domain dictated by (2). The time independent approach adopted in this book proceeds by deriving two spectral representations for the operator B. In the present case, this boils down to proving eigenfunction expansion theorems in $L_2(\mathbf{R}^n \setminus \Gamma)$ for the selfadjoint operator, A, given by the Laplacian with Neumann boundary conditions. When Γ is the empty set this expansion is the usual L_2 theory of the Fourier transform with eigenfunctions $e^{ix\xi}$, $\xi \in \mathbf{R}^n$. For nonempty Γ the eigenfunctions are of the form $e^{ix\xi} + O(1/|x|)$; in particular, they are not square integrable. The proof of the expansion theorem has as its starting point the formula of Stone for the spectral projections of a selfadjoint operator, A,

$$(E_b - E_a)f = \frac{1}{2\pi i} \lim_{\varepsilon \to 0} \int_a^b [R(x + i\varepsilon) - R(x - i\varepsilon)] f \, dx$$

where $R(z) = (z - A)^{-1}$ and the points a and b are not eigenvalues of A. The main effort is in the study of the behavior of $R(x \pm i\varepsilon)f$ as $\varepsilon \to 0$. Since we are interested in the case where $x \in \sigma(A)$ these limits will not exist in $L_2(\mathbf{R}^n \setminus \Gamma)$, however, one gets convergence in $L_2^{\mathrm{loc}}(\mathbf{R}^n \setminus \Gamma)$.

Once the eigenfunction expansions are obtained the existence and completeness of the wave operators is proved and then, in a final chapter, the implications for the transport of energy are investigated. Specifically, a formula is obtained for the energy which is scattered into a conical region in $\mathbf{R}^n$. This last chapter is a welcome return to the concrete questions which motivated the theory. Too often big theorems are proved without standing back afterward to see what has been accomplished.

Wilcox' book is written with an attention to detail which is unusual in the literature on partial differential equations. Only a knowledge of integration and the spectral theorem for unbounded operators is assumed and the text could be used by second year graduate students with this background. There are several well-chosen heuristic arguments to clarify the exposition. Unfortunately, there are some parts of the theory which seem to be hard to motivate. In particular, the connection between the $\pm$ in $R(x\pm i\varepsilon)$ and the radiation conditions of Sommerfeld, and, the existence of *two* special spectral representations seem to arise mysteriously. Overall the book is easy to read, has very few typographical errors, and is quite informative. It is, I think, the best introduction to scattering theory that is available. The reader should be warned, however, of two weaknesses. First, there are several places where crucial results are presented with references rather than proofs, and second, there is a tendency to overwrite some simple proofs. The impact is to skew the presentation in the direction of verifying details at the expense of some beautiful and hard analysis. On the other hand a serious reader who consults the references supplied by the author (or those suggested below) can obtain a balanced and complete presentation.

The important omissions are the following:

1. To obtain the asymptotic behavior of solutions (pp. 23ff) of the free wave equation the method of stationary phase is used but not proved. A clean proof is given in §1.2 of [**1**].

2. Rellich's uniqueness theorem is not proved.

3. In the proof of local decay of solutions (p. 81) an abstract result is quoted with reference giving the misimpression that a difficult result is being used. The proof requires only a few lines.

4. To show that the functions $\lim_{\varepsilon\to 0} R(x\pm i\varepsilon)f$ satisfy radiation conditions (pp. 72–73) one needs to know the asymptotic behavior of the free space Green's function. The need for and proof of this information is not indicated in the text. The appropriate facts can be found on p. 127 of [**2**].

An example where a simple proof is given in so much detail that the simplicity is obscured is the density argument on p. 139. A single well-chosen sentence would suffice as in a similar context on the bottom of p. 127. A second example is the proof of Lemma 6.3 which could be given in about one half the space.

One of the author's innovations is that the theory is carried out without strong smoothness hypotheses on the obstacle. Only Rellich's compactness theorem in $\mathbf{R}^n\backslash\Gamma$ is needed and not regularity up to the boundary of solutions to the Neumann problem in $\mathbf{R}^n\backslash\Gamma$. This depends essentially on the fact that the Neumann condition can be given a weak or variational formulation. Though this observation is no news to experts it is good to see it in print. It is also worth noting that the standard boundary value problems for Maxwell's equations can also be cast in variational form [**3**, pp. 284–285] and therefore the scattering theory is valid for non-smooth obstacles.

Let me close by saying that reading this book was both pleasurable and informative and I hope that this experience is shared by many mathematicians.

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Latin squares and their applications, by J. Dénes and A. D. Keedwell, Academic Press, New York, 1974, 547 pp., $24.50.

A *latin square* $A=[a_{ij}]$ of order n is an $n\times n$ array in which the places are occupied by elements from an n-element set and each element from the set occurs exactly once in each row and column. They are familiar objects in algebra as multiplication tables of quasigroups, in geometry as coordinate systems for nets, and in statistics where, as one of the simplest combinatorial designs, they are used extensively in the design of experiments.

This is the first book devoted entirely to latin squares. While the statistical, algebraic and geometric aspects are discussed, the major theme is the construction of orthogonal sets of latin squares. This is not surprising since much of the current interest in latin squares was stimulated by the disposal in the late 1950's of a famous conjecture of Euler's. Two $n\times n$ latin squares $A=[a_{ij}]$, $B=[b_{ij}]$ are *orthogonal* if, when B is superimposed on A, the n^2 ordered pairs (a_{ij}, b_{ij}) contain each pair exactly once. Euler's *Officers Problem* concerns the existence of a 6×6 array of 36 officers, 6 of each rank, from 6 different regiments, such that there is, in each row and in each column, exactly one officer of each rank and one officer from each regiment. This is obviously equivalent to the existence of two orthogonal latin squares of order six. Euler was able to construct a pair of orthogonal latin squares for all orders n other than $n\equiv 2 \pmod 4$ and he conjectured that for these orders no such pair exists. That Euler's conjecture is true for $n=6$ was verified by Tarry in 1900. It was not until 1958–1960 that the combined efforts of Bose, Shrikhande and Parker showed that Euler was wrong in all other cases.

At about the same time, another well-known conjecture was disposed of by Parker. Macneish in 1922 conjectured that if $n=p_1^{i_1}p_2^{i_2}\cdots p_m^{i_m}$ (p_i distinct primes) then the maximal size of a set of mutually orthogonal latin squares (m.o.l.s.) is $(\min p_j^{p_j})-1$. This conjecture is based on the construction of a set of p^i-1 m.o.l.s. from a finite field of order p^i. By a direct product construction we can obtain from t m.o.l.s. of order n_1 and t m.o.l.s. of order n_2, a set of t m.o.l.s. of order n_1n_2. Thus, for $n=p_1^{i_1}p_2^{i_2}\cdots p_m^{i_m}$, there is a set of at least $(\min p_j^{i_j})-1$ m.o.l.s. Macneish conjectured that there were exactly this many. However, Parker constructed a set of 4 m.o.l.s. of order 21. More recently, sets of 5 m.o.l.s. of order 12 have been constructed. Since we now know that there do exist sets of m.o.l.s. for all $n>6$, interest has shifted to the question of the maximal size of such sets. We know that the number tends to infinity