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ON SCARCITY OF OPERATORS WITH FINITE SPECTRUM

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Let ρ denote the spectral radius of an operator; in 1968–1970 Edoardo Vesentini proved

PROPOSITION 1 ([5] AND [6]). *If $\lambda \longrightarrow f(\lambda)$ is an analytic function mapping a domain in* $\mathbf{C}$ *into a complex Banach algebra then $\lambda \longrightarrow \operatorname{Log} \rho(f(\lambda))$ is subharmonic.*

From this we got the following generalization of Newburgh's continuity theorem [4], where $\sigma(x)$ is the union of Sp x and its holes.

PROPOSITION 2 (ALMOST-CONTINUITY THEOREM). *If $\lambda \longrightarrow f(\lambda)$ is analytic on a domain D containing z_0 and if E is a subset of D, such that $z_0 \in \overline{E}$, E is nonsharp at z_0, then there exists a sequence (λ_n) converging to z_0 with $\lambda_n \in E$, $\lambda_n \neq z_0$, and $\lim_{n\to\infty} \sigma(f(\lambda_n)) = \sigma(f(\lambda_0))$.*

The same statement with the spectrum is false. If δ is the diameter of the spectrum, we obtained as well

PROPOSITION 3. *With the same hypothesis $\lambda \longrightarrow \operatorname{Log} \delta(f(\lambda))$ is subharmonic.*

All of these results and intricate properties of subharmonic functions, capacity and sharp sets, easily found in [1], give the fundamental theorem

THEOREM 1. *Either the set of λ, such that the spectrum of $f(\lambda)$ is finite, is of outer capacity zero, or there exists an integer n such that the spectrum of $f(\lambda)$ has exactly n elements, for every λ, except on a closed set of capacity zero, where the spectrum has at most $n - 1$ elements.*

Kaplansky [3], in 1954, and Hirschfeld-Johnson [2], in 1972, proved that $A/\text{Rad } A$ is finite dimensional, for a complex Banach algebra A, if the spectrum of every element of this algebra is finite. Unfortunately the method does not work for local and real cases. Other persons (Behncke, Wong) obtained the same result for A^*-algebras supposing the spectrum finite for hermitian elements.

Theorem 1 can be used to get

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THEOREM 2. *Let A be a complex Banach algebra, H a closed real subspace of A such that $A = H + iH$. If there exists a nonempty open set U of H such that $x \in U$ implies* $\mathrm{Sp}\, x$ *finite, then $A/\mathrm{Rad}\, A$ is finite dimensional.*

And particularly

THEOREM 3. *Let A be a real Banach algebra containing a nonempty open set U such that $x \in U$ implies* $\mathrm{Sp}\, x$ *finite; then $A/\mathrm{Rad}\, A$ is finite dimensional.*

THEOREM 4. *Let A be a complex Banach algebra with involution such that the set of hermitian elements contains a nonempty open set U with the property $x \in U$ implies* $\mathrm{Sp}\, x$ *finite; then $A/\mathrm{Rad}\, A$ is finite dimensional.*

Full details will appear elsewhere.

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