BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 3, May 1976

BLASCHKE PRODUCTS GENERATE H^∞

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Communicated by Robert G. Bartle, January 26, 1976

1. **Introduction.** Let H^∞ be the algebra of bounded analytic functions on the unit disc Δ in the complex plane. By Fatou's theorem, every function $f \in H^\infty$ has a nontangential limit $f(e^{i\theta})$ almost everywhere on $\partial\Delta$. By identifying each H^∞ function with its boundary-value function, we can view H^∞ as a uniformly closed subalgebra of $L^\infty(\partial\Delta, d\theta/2\pi)$. An inner function is a function $u \in H^\infty$ such that $|u(e^{i\theta})| = 1$ almost everywhere. Let J be the smallest uniformly closed subalgebra of H^∞ containing all inner functions. In [2] and [4], the problem of identifying J arose.

2. **Main result.** A Blaschke product b is an inner function of the form

$$b(z) = c \prod_{n=1}^{\infty} \frac{|z_n|}{-z_n} \left(\frac{z - z_n}{1 - \bar{z}_n z} \right)$$

where $|c| = 1$, $z_n \in \Delta$ and $\Sigma(1 - |z_n|) < \infty$. Frostman [9] has shown that every inner function can be uniformly approximated by Blaschke products. Carathéodory [3] has shown that every H^∞ function with norm ≤ 1 can be approximated uniformly on compact subsets of Δ with finite Blaschke products. The following theorem can be viewed as a generalization of his result.

THEOREM 1. *Finite linear combinations of Blaschke products are uniformly dense in H^∞.*

To accomplish the proof, we introduce an auxiliary subalgebra of H^∞. We let

$$N = \{ f \in H^\infty : \bar{f} u \in H^\infty \text{ for some inner function } u \}.$$

The reason for the terminology is that a function $f \in H^\infty$ is in N if and only if $f(e^{i\theta})$ is the boundary-value function of a function from the Nevanlinna class on $\{z: |z| > 1\}$. In other words, each $f \in N$ has a "pseudocontinuation" to the whole Riemann sphere. Indeed if $f \in N$ and $\bar{f} u = g \in H^\infty$, then

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AMS (MOS) subject classifications (1970). Primary 30A76, 30A78, 30A82; Secondary 46J10, 46J15.

Key words and phrases. Blaschke products, inner functions, bounded analytic functions, H^∞, uniform approximation, pseudocontinuation, uniform algebras, convex hull.

[1] This paper was supported in part by National Science Foundation Grant MPS 74-7035.

$$f(e^{i\theta}) = \lim_{r\to 1^+} \overline{g(1/\bar{z})/u(1/\bar{z})} \qquad z = re^{i\theta} \text{ (a.e.)}.$$

To see the converse, notice that there is an $h \in H^2$ such that $\bar{f}h \in H^2$. The set of all such functions h is a closed subspace of H^2 invariant under the shift operator. By Beurling's theorem, it contains an inner function. These observations were first made in [5].

Alain Bernard has shown that N is a dense subalgebra of J. The proof runs as follows: It is clear that N is an algebra. Also if $u_1, \ldots, u_n$ are inner functions and $f = \Sigma\lambda_i u_i$, $\lambda_i \in \mathbf{C}$, then setting $u = \Pi u_i$, we see that $\bar{f}u \in H^\infty$. Now if $f \in H^\infty$ with $\bar{f}u \in H^\infty$ and $\|f\| < 1$, then for all real t

$$f_t = (f + ue^{it})/(1 + \bar{f}ue^{it}) \in H^\infty$$

and f_t is an inner function. But

$$f = \frac{1}{2\pi}\int_0^{2\pi} f_t \, dt$$

and as $\|f\| < 1$, the integral converges uniformly on $[0, 2\pi]$. Some Riemann sums now give an approximation to f by convex combinations of inner functions.

To prove Theorem 1, let $f \in H^\infty$. Douglas and Rudin [4] have shown that for every $\epsilon > 0$ there exist Blaschke products $b_0, \ldots, b_n$ and $\lambda_i \in \mathbf{C}$ such that

$$\left\| f - \left(\sum_{i=1}^{n} \lambda_i b_i\right)\Big/ b_0 \right\| < \epsilon.$$

Let $g = \Sigma_{i=1}^n \lambda_i b_i$. Then the coset $-g/\epsilon + b_0 H^\infty$ has norm < 1. By Nevanlinna's theorem, Satz 7 of [10], there is an inner function v such that $\epsilon v = -g + b_0 h$, for some $h \in H^\infty$. Now $v \in N$ and $g \in N$ and N is an algebra, so there is an inner function u with $\overline{b_0 h}u \in H^\infty$. Notice that $\bar{h}u = \overline{b_0 h}u \cdot b_0 \in H^\infty$, so that $h \in N$. Finally,

$$\|h - f\| = \|b_0 h - b_0 f\| \leqslant \|b_0 h - g\| + \|g - b_0 f\| < 2\epsilon,$$

proving Theorem 1.

COROLLARY. *The set of H^∞ functions which have a pseudocontinuation to $\{z\colon |z| > 1\}$ are uniformly dense in H^∞.*

Bernard used his idea to prove the following general theorem, which contains results of Phelps [11], Sine [13], Fisher [6], [7], [8], and Rudin [12].

THEOREM (BERNARD). *If A is a uniform algebra generated by unimodular functions, then the closed unit ball of A is the norm-closed convex hull of the unimodular functions in A.*

COROLLARY. *The norm-closed convex hull of the Blaschke products is the closed unit ball of H^∞.*

In [1], similar results will be proved for more general domains than the unit disc and for weak-$*$ closed logmodular uniform algebras.

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