BOOK REVIEWS

Topological vector spaces, by A. Grothendieck, Gordon and Breach, New York,
London, Paris, 1973, x + 245 pp., $19.50. (Translated by Orlando Chaljub)

This book is a translation of the notes of a course given by Grothendieck in
São Paulo in 1954 and published in French during the same year (a second
edition came out in 1958, a third in 1964). The original lecture notes had
several distinctions: they constituted the first expository treatment of locally
convex spaces, they contained much material which could not be found
anywhere else, and, most importantly, they were written by a man who had
become one of the leading mathematicians of the 20th century.

Grothendieck was engaged in research on topological vector spaces between
1950 and 1953. During this period he put his stamp on the theory and proved
some of its deepest results. In 1954 he wrote his lectures in complete mastery
of the whole field as the top expert on the subject: already for this reason alone
the notes deserve to appear as a printed book.

It begins with a Chapter 0 entitled "Topological introduction" and contain-
ing preliminary material concerning topics which were less well known in 1954
than they are now: initial and final topologies, precompact sets, topologies in
function spaces and equicontinuity. The writing is very concise, and since
Chapters I–IV, VIII and IX of Bourbaki's "Topologie générale" are stated to
be prerequisites anyway, the reader is advised to study the material in
Bourbaki's Chapter X, preferably in the "entirely recast" 1961 edition.

Very little of Chapter 0 is used later on, so that one can start with Chapter
1: "General properties", where in 32 pages one gets an elegant introduction to
the basic concepts: topologies compatible with vector structure, subspaces,
quotients, products, direct sums, continuous linear maps, bounded sets, $\mathfrak{S}$-
topologies (which in the translation became $\mathfrak{G}$-topologies). Locally convex
spaces are defined through seminorms, and since convex sets appear only in
Chapter 2, their characterization as spaces with a fundamental system of
convex neighborhoods is given there. Chapter 1 contains the Banach homo-
morphism theorem, the closed graph theorem and the Banach-Steinhaus
theorem, with their usual corollaries.

The title of Chapter 2 is "The general duality theorems on locally convex
spaces". After a discussion of convex sets, the geometric form of the Hahn-
Banach theorem is proved in a simple way. There exist now several proofs of
the analytic form, one based on Banach's original idea, one on a theorem
concerning the extension of positive linear forms. Aumann and Dinges
considered a generalization of the latter, in which vector spaces are replaced
by commutative monoids, the best result in this direction is due to Scheller
[33]. There is a relation between extensions of additive maps and "sandwich"
theorems which affirm the existence of an additive map between a subadditive
and a superadditive one [12], [22].

From the geometric form of the Hahn-Banach theorem Grothendieck deduces the analytic one, and the usual consequences of the two. Vector spaces in duality are introduced, together with the accompanying concepts: weak topology, polarity, $\mathfrak{S}$-topologies. If (E, E') is a dual system and $\mathfrak{S}$ a collection of bounded subsets of E, the dual of E' equipped with the $\mathfrak{S}$-topology is determined: this leads to criteria for reflexivity and the Mackey-Arens theorem characterizing the topologies compatible with the duality. Next Grothendieck proves his own completeness theorem. In §16 he studies transposes of linear maps and topological homomorphisms (= strict morphisms). He proves that $u: E \to F$ is a strict morphism if and only if:

(i) the image $u(F')$ of F' by the transpose $^t u: F' \to E'$ is $\sigma(E', E)$-closed (i.e., weak*-closed),

(ii) every equicontinuous subset E' contained in $^t u(F')$ is the image by $^t u$ of an equicontinuous subset of F'.

Köthe [23] observed that (ii) holds exactly if u is almost open, i.e., for every neighborhood V of 0 in E the closure of $u(V)$ is a neighborhood of 0 in $\overline{u(E)}$. Any linear map from a locally convex Hausdorff space into a barrelled space is almost open. Denote by $\nu(E', E)$ the finest topology on E' which induces on every equicontinuous set the same topology as $\sigma(E', E)$; it is introduced on p. 159 of the book under review. If every $\nu(E', E)$-closed subspace of E' is $\sigma(E', E)$-closed then E is said to be a Pták (or B-complete or fully complete) space. It follows from Köthe's observation that any continuous, almost open linear map from a Pták space onto a locally convex Hausdorff space is a strict morphism and so we get Pták's homomorphism theorem (Exercise 1 on pp. 160–161 of Grothendieck's book): any surjective, continuous, linear map from a Pták space onto a barrelled space is a strict morphism.

If every $\nu(E', E)$-closed, $\sigma(E', E)$-dense subspace of E' is equal to E', then E is an infra-Pták (B_r-complete) space. Pták's closed graph theorem states that every linear map from a barrelled space into an infra-Pták space, whose graph is closed, is continuous. Adasch, Komura and Valdívia [19] have determined the largest class of spaces–the "infra-s-spaces"–such that every closed linear map from a barrelled space into a space of the class is continuous. Other maximal classes of source and target spaces have been determined [8], [9], [11], [19] and their properties investigated in view of the two outstanding conjectures: Is every infra-Pták space a Pták space? Does there exist on any infra-s-space a finer infra-Pták topology? The conjecture that for every infra-s topology the associated barrelled topology is infra-Pták has been disproved [10].

The last paragraph, §18 of Chapter 2, presents elementary properties of compact sets and compact maps and introduces Montel spaces.

Chapter 3 is devoted to "Spaces of linear mappings". After $\mathfrak{S}$-topologies and bounded sets, Grothendieck defines barrelled, quasi-barrelled (= infra-barrelled) and bornological spaces. When these spaces were introduced, it was an open problem whether there exist barrelled spaces which are not bornological. The first examples were given in 1954 by Nachbin and Shirota; since then many more were found [42]. We also know that a barrelled, bornological space is not necessarily ultrabornological (called strictly bornological in the book, p. 148) [40], and that the space E of §3, Exercise 8 (p. 109) is such an example [1]. Every Baire space is barrelled, but, as the same exercise shows, the

converse is false. Recently a whole scale of spaces has been intercalated between Baire and barrelled spaces [30], [31], [32].

Next Grothendieck introduces bilinear maps, hypocontinuity and spaces of bilinear maps. The ε-tensor-product makes a brief appearance in §6, pp. 124–126. The chapter ends with two examples: spaces of continuous linear functions from a locally convex space into a space of continuous functions, and differentiable functions with values in a locally convex space.

Chapter 4 has the title "Study of some special classes of spaces" and consists of four parts, the first of which is called "Inductive limits, (ℒℱ) spaces". After some basic definitions, facts and examples, strict inductive limits, direct sums and, in particular, direct sums and products of lines are taken up. We now know [7] that the answer to the question asked on p. 140, after the definition of a strict inductive limit, is negative: there exist strict inductive limits E of uncountable families $(E_i$ of locally convex spaces such that not every bounded subset of E is contained in one of the E_i.

An (ℒℱ) space E is defined as the inductive limit of a sequence E_i of Fréchet spaces with respect to linear maps $u_i: E_i \to E$ such that $\cup u_i(E_i)$ generates E. Here Grothendieck proves his closed graph theorem: let E be an ultrabornological space and F a locally convex space which is the union of a sequence of images of Fréchet spaces by continuous linear maps; then every linear map $E \to F$, whose graph is sequentially closed, is continuous. Then he adds: "It seems we could considerably weaken the conditions on F, a question worth some research." I am happy to report that this research has been brilliantly performed by Marc De Wilde [3]; [6]. A web on a vector space E is a family $\mathfrak{W} = \{C(n_1, \ldots, n_k)\}$ of subsets of E, where $k, n_1, n_2, \ldots$ are integers $\geqslant 1$, such that

$$E = \bigcup_{n_1=1}^{\infty} C(n_1), \ldots, C(n_1, \ldots, n_{k-1}) = \bigcup_{n_k=1}^{\infty} C(n_1, \ldots, n_{k-1}, n_k).$$

A web $\mathfrak{W}$ on a locally convex Hausdorff space E is of type $\mathcal{C}$ if for every sequence (n_k) there exist $\rho_k > 0$ such that the conditions $0 \leqslant \lambda_k \leqslant \rho_k$ and $x_k \in C(n_1, \ldots, n_k)$ for $k \geqslant 1$ imply that $\sum_{k=1}^{\infty} \lambda_k x_k$ converges in E. De Wilde's closed graph theorem can be stated as follows: any linear map with sequentially closed graph, from an inductive limit of Fréchet spaces into a locally convex space with a web of type $\mathcal{C}$, is continuous. Spaces with webs of type $\mathcal{C}$ have remarkable stability properties, satisfying all the desiderata of Grothendieck [17, Introduction IV, pp. 18–19].

Grothendieck was the first to observe that a subspace of an (ℒℱ) space is not necessarily an (ℒℱ) space and Kascic and Roth gave an example of such a subspace in the Schwartz test function space $\mathcal{D}(\Omega)$. This led to the concept of well-located subspace of an (ℒℱ) space [6].

Part 2 of Chapter 4 is on metrizable locally convex spaces. The main result is the Banach-Dieudonné theorem according to which on the dual E' of a metrizable space E the topology $\nu(E', E)$ is the topology of uniform convergence on precompact subsets of E. In §4, p. 162 (whose heading got lost in the translation) homomorphisms from one Fréchet space into another are characterized following Dieudonné-Schwartz, and one obtains a condition used in the theory of partial differential equations [36]: $u: E \to F$ is surjective if and

only if $'u$ is injective and $'u(F')$ is $\sigma(F', F)$-closed. Several extensions of this condition were given [18], [35].

To define elegantly the topic of Part 3, "($\mathcal{DF}$) spaces", one best introduces some variants of the notion of barrelledness, which have been studied lately [20], [4]. A locally convex Hausdorff space is countably barrelled if every weak* bounded subset of the dual, which is the union of a sequence of equicontinuous subsets, is equicontinuous; it is sequentially barrelled if every weak* bounded sequence is equicontinuous. The concepts of countable and sequential infra-barrelledness are obtained replacing weak* boundedness by strong boundedness. The strong dual of a Fréchet space is not necessarily infra-barrelled but Grothendieck [16] observed that it is always countably infra-barrelled. Motivated by this observation, he defined a ($\mathcal{DF}$) space as a countably infra-barrelled space which possesses a countable fundamental system of bounded sets. The present book contains only a selection of his results, for more details he refers to [16].

The connection of the various barrelledness concepts with the behavior of balanced, convex sets was investigated by De Wilde and Houet [4], [5], who generalized, in particular, a result of Valdívia [38], from which it follows that a countable-codimensional subspace of a barrelled space is itself barrelled. This result was also proved by Levin and Saxon [24], for finite codimension it is due to Dieudonné. Similarly, a finite-codimensional subspace of an infra-barrelled (resp. bornological) space is infra-barrelled (resp. bornological) [37]. On the other hand a countable-codimensional subspace of a bornological space is not necessarily even infra-barrelled [41], but a countable-codimensional subspace of an ultrabornological space is bornological [39]. Other generalizations of barrelledness were investigated by Marquina and Pérez Carreras [26].

Part 4 deals with "Quasi-normable and Schwartz spaces", also introduced in [16]. Schwartz spaces are nowadays mostly considered in terms of operator-ideals of Pietsch, which permits studying them simultaneously with other classes of spaces, e.g., nuclear ones [21]. Schwartz maps, the analogs of nuclear maps, were defined by Randtke [27].

Chapter 5, "Compactness in locally convex topological vector spaces", is again divided into four parts. The first of these is concerned with "The Krein-Milman theorem", according to which a compact, convex subset K of a locally convex space is the closed, convex hull of the set $Ep(K)$ of its extreme points. Choquet has made the theorem more precise by proving that every $x \in K$ is the barycenter of a measure, which–if K is metrizable–is carried by $Ep(K)$, and defined a simplex as a set for which this representation is unique. Choquet's theory is one of the most important contributions to functional analysis since Grothendieck's book was written.

Part 2 is the "Theory of compact operators". It treats operators of the form $u + v$, where u is an isomorphism and v a compact map, following a note of Laurent Schwartz [34], and the Riesz theory of operators of the form $I + v$. This theory has been extensively studied and refined in recent years [2].

Visibly the topics treated in the last two parts of Chapter 5 were close to Grothendieck's heart. They are "General criteria of compactness", associated with the names of Šmulian, Eberlein and Krein, and the Dunford-Pettis criterion for "Weak compactness in L^1". On each of these topics he has written

a paper [14], [15] and he lists 47 exercises, which take up almost 24 of the 40 pages devoted to the two parts. Much of the material presented in the exercises cannot be found anywhere else.

Even after twenty years, Grothendieck's book is an elegant and refreshing introduction to topological vector spaces, and in spite of the fact that at least ten monographs have been written on the subject since 1954, it is probably the best textbook to use in a course. The proofs are at times concise or even omitted, but this enhances its value as a textbook. An additional feature is the 195 exercises, some of them quite challenging, which cover about 61 of the book's 245 pages.

The translation has scrupulously reproduced the numerous misprints of the original and added generously on its own. Exercise 1 on p. 33 was taken over into the English text in spite of the fact that in the new edition of [17] Grothendieck points out that it is false. But the trouble with the translation goes deeper. When Grothendieck's notes were written, the terms for injective and surjective were one-to-one (biunivoque) and onto (sur). However, "biunivoque" is translated everywhere by "bijective" (e.g., p. 75; Exercise 1, p. 152; p. 198), which leads to the following gem: " ... $\tilde{v}$ is clearly bijective ... we shall show that it is *onto*" (p. 197). Frequently the original had to be consulted in order to understand the meaning of a sentence. Take for instance the proof of Corollary 2 of Theorem 3 on p. 57. In the original text this reads: "On est en effet ramené à prouver que si V est un sous-espace vectoriel fermé, et $x \in CV$, il existe un hyperplan fermé contenant V et non x, ce qui résulte aussitôt du th. 3" and has been rendered as follows: "We can prove that if V is a closed vector space, and $x \in CV$, there exists a closed hyperplane containing V and not x, which is a result of Theorem 3." Or on p. 142 "This linear form is restricted to the equicontinuous subsets of E_i' which are weakly continuous" is offered as a translation of "Cette forme linéaire a ses restrictions aux parties équicontinues de $(E_i)'$ faiblement continues." It makes a difference whether a cone contains 0 or has 0 as its vertex (p. 188). On p. 222 "pas tout à fait" is translated as "not at all". It would be easy but futile to go on with these examples. Words and even sentences are missing: on p. 81, in the statement of Proposition 25, instead of "A a subset of F'", it should say "A a subset of E, B a subset of F'"; on p. 110 the words "is bounded" must be added to the penultimate sentence of the exercise; on p. 130 for "coordinates $i \in CJ$" read "coordinates with indices $i \in CJ$"; on p. 135 a remark and three exercises were simply omitted; on p. 240, line 5 after $\partial T/\partial z_i = 0$ the words "for every i, hence by a classical theorem of Schwartz, it is defined" are missing. The cross references have not been checked; there is, for instance, no Exercise 3 in Chapter 2, §3 (p. 114).

There is no terminological index which is the more bothersome as some concepts are mentioned before they are formally defined. Thus quasi-complete spaces appear on p. 96 and are defined on p. 104; ($\mathcal{F}$) spaces are never defined, but their definition is "recalled" on 154. There is, of course, no bibliography.

Obviously the interspersed remarks made in this review represent only a small and subjective cross-section of the progress of the theory of locally convex spaces since 1954, many important topics have not been mentioned. Similarly, the list below contains only a tiny portion of the publications on the subject.

References

1. J. Batt, P. Dierolf and J. Voigt, *Summable sequences and topological properties of* $m_0(I)$ (to appear).

2. S. R. Caradus, W. E. Pfaffenberger and B. Yood, *Calkin algebras and algebras of operators on Banach spaces*, Lecture Notes in Pure and Appl. Math., vol. 9, Dekker, New York, 1974.

3. M. De Wilde, *Réseaux dans les espaces linéaires à semi-normes*, Mém. Soc. Roy. Sci. Liège Coll. in-8° (5) **18** (1969), no. 2, 144 pp. MR **44** #3102.

4. ———, *Various types of barrelledness and increasing sequences of balanced and convex sets in locally convex spaces*, Lecture Notes in Math., vol. 331, Springer-Verlag, Berlin, Heidelberg, New York, 1973, pp. 211–217.

5. M. De Wilde and C. Houet, *On increasing sequences of absolutely convex sets in locally convex spaces*, Math. Ann. **192** (1971), 257–261. MR **44** #5746.

6. M. Dostal, *Some recent results on topological vector spaces*, Lecture Notes in Math., vol. 384, Springer-Verlag, Berlin, Heidelberg, New York, 1974, pp. 21–97.

7. A. Douady, *Parties compacte d'un espace de fonctions continues à support compact*, C. R. Acad. Sci. Paris **257** (1963), 2788–2791. MR **27** #6121.

8. V. Eberhardt, *Durch Graphensätze definierte lokalkonvexe Räume*, Thesis, München, 1972.

9. ———, *Über einen Graphensatz für Abbildungen mit normiertem Zielraum*, Manuscripta Math. **12** (1974), 47–65. MR **49** #1056.

10. ———, *Einige Vererbbarkeitseigenschaften von B- und B_r-vollständigen Räumen*, Math. Ann. **215** (1975), 1–11.

11. V. Eberhardt and W. Roelcke, *Über einen Graphensatz für lineare Abbildungen mit metrisierbarem Zielraum*, Manuscripta Math. **13** (1974), 53–68.

12. B. Fuchssteiner, *Sandwich theorems and lattice semigroups*, J. Functional Analysis **16** (1974), 1–14. MR **49** #10619.

13. G. B. Giráldez Tiebo, *Teoremas de gráfico cerrado, nucleo cerrado y aplicación abierta*, Thesis, Madrid, 1974.

14. A. Grothendieck, *Critères de compacité dans les espaces fonctionnels généraux*, Amer. J. Math. **74** (1952), 168–186. MR **13**, 857.

15. ———, *Sur les applications linéaires faiblement compactes d'espaces du type C(K)*, Canad. J. Math. **5** (1953), 129–173. MR **15**, 438.

16. ———, *Sur les espaces (𝔉) et (𝔇𝔉)*, Summa Brasil. Math. **3** (1954), 57–123. MR **17**, 765.

17. ———, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. No. 16 (1955). MR **17**, 763.

18. C. Harvey, *On domination estimates and global existence*, J. Math. Mech. **16** (1967), 675–702. MR **35** #5750.

19. J. Horváth, *Quelques résultats de M. H. Powell sur le théorème du graphe fermé*, Publ. Dép. Math. (Lyon) **10** (1973), 65–77. MR **51** #1330.

20. T. Husain, *Two new classes of locally convex spaces*, Math. Ann. **166** (1966), 289–299. MR **35** #3406.

21. H. Jarchow, *Nuclear locally convex spaces*, Lecture notes, no. 13, University of Maryland, 1976.

22. P. Jimenez Guerra, Thesis, Madrid, 1974.

23. G. Köthe, *General linear transformations of locally convex spaces*, Math. Ann. **159** (1965), 309–328. MR **33** #573.

24. M. Levin and S. Saxon, *Every countable-codimensional subspace of a barrelled space is barrelled*, Proc. Amer. Math. Soc. **29** (1971), 91–96. MR **43** #6691.

25. A. Marquina and P. Pérez Carreras, *On quasibarrelled spaces*, Manuscripta Math. **12** (1974), 387–398.

26. ———, *Other results in 𝔊𝔅-barrelled spaces*, Proc. Sympos. Functional Analysis, Istanbul, Silivri, 1973, pp. 115–116.

27. D. J. Randtke, *Characterization of precompact maps, Schwartz spaces and nuclear spaces*, Trans. Amer. Math. Soc. **165** (1972), 87–101. MR **46** #4139.

28. W. Robertson, *On the closed graph theorem and spaces with webs*, Proc. London Math. Soc. (3) **24** (1972), 692–738. MR **46** #5979.

29. ———, *Sur le théorème du graphe fermé*, Bull. Soc. Math. France Suppl. Mém. 31/32 (1972), 343–350.

30. S. A. Saxon, *Nuclear and product spaces, Baire-like spaces, and the strongest locally convex topology*, Math. Ann. **197** (1972), 87–106. MR **46** #4140.

31. ———, *Some normed barrelled spaces which are not Baire*, Math. Ann. **209** (1974), 153–160.

32. S. A. Saxon and A. R. Todd, *A property of locally convex Baire spaces*, Math. Ann. **206** (1973), 23–34. MR **48** #6865.

33. L. Scheller, *Fortsetzungen additiver Funktionen auf Halbgruppen zwischen sub- und superadditiven Funktionalen*, Thesis, Frankfurt am Main, 1973.

34. L. Schwartz, *Homomorphismes et applications complètement continues*, C. R. Acad. Sci. Paris **236** (1953), 2472–2473. MR **15**, 233.

35. F. Trèves, *Locally convex spaces and linear partial differential equations* Die Grundlehren der math. Wissenschaften, Band 146, Springer-Verlag, New York, 1967. MR **36** #6986.

36. ———, *Applications of distributions to PDE theory*, Amer. Math. Monthly **77** (1970), 241–248. MR **40** #6035.

37. M. Valdívia, *A hereditary property in locally convex spaces*, Ann. Inst. Fourier (Grenoble) **21** (1972), fasc. 2, 1–2. MR **48** #11967.

38. ———, *Absolutely convex sets in barrelled spaces*, Ann. Inst. Fourier (Grenoble) **21** (1972), fasc. 2, 3–13. MR **48** #11968.

39. ———, *On final topologies*, J. Reine Angew. Math. **251** (1971), 193–199. MR **45** #4108.

40. ———, *A class of bornological barrelled spaces which are not ultrabornological*, Math. Ann. **194** (1971), 43–51. MR **47** #2307.

41. ———, *Some examples on quasi-barrelled spaces*, Ann. Inst. Fourier (Grenoble) **22** (1972), fasc. 2, 21–26. MR **49** #1053.

42. ———, *On nonbornological barrelled spaces*, Ann. Inst. Fourier (Grenoble) **22** (1972), fasc. 2, 27–30. MR **49** #1050.

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Partial differential equations, by E. T. Copson, Cambridge University Press, Cambridge (England), London, New York, Melbourne, 1975, vii + 280 pp.

Many basic laws of nature can be formulated as systems of differential equations, ordinary or partial. Predictions of physical phenomena then present themselves as boundary problems for such systems. Many of them are formidable mathematical challenges not yet mastered. Those which have been solved have required the entire arsenal of analysis, power series, separation of variables, successive approximations, Fourier analysis, functional analysis and distributions. On the other hand, most of these tools were created to solve problems in physics. The classical linear partial differential equations are Laplace's equation of potential theory, the wave equation of the theory of wave propagation, and the heat equation of the theory of heat conduction. The diversity of the physics involved explains the fact that the corresponding boundary problems are quite different and also the methods for their solution.

Riemann's lectures, *Partial differential equations and their applications*, published by Hallendorff in 1882, was the first systematic book in the field. Twenty years later came an expanded version by Weber, which after another twenty years branched out into the encyclopedic *Differential equations of physics* by Frank and von Mises. At about the same time, *Methods of mathematical physics* by Courant and Hilbert, and Webster's *Partial differential equations of mathematical physics* made their appearance. The aim was twofold: to give old and new mathematical tools to physicists and to introduce