changing the level of the book, but I do not want to go into the details of this. Instead I should like to make a remark that may be of general interest.

Copson's book is full of examples from physics but they are not presented in a very systematic way. Sometimes they just illustrate the algebraic classification into types, sometimes they motivate the classification. I believe that physics should have the lead and this for two reasons. A full physical introduction to, e.g., wave propagation gives the reader in one stroke many intuitive aspects that will appear later in the mathematical theory. Also, when the physicists turned to quantum mechanics, they left classical physics, heat, electricity, hydrodynamics and waves to the engineers, who, unfortunately, have to specialize in one of these branches. The mathematicians are the only ones who now have the opportunity of giving students something like a general education in classical physics. This opportunity should be used. Wave propagation, potential theory and heat conduction should appear as early as reasonably possible in every standard calculus course. The same goes for that indispensable tool, the Fourier transform. Neglecting these simple applications, calculus is not really a serious affair and–to use Webster's words–its lofty aim may get out of sight.

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Elliptic functions, by Serge Lang, Addison Wesley, Reading, Mass., 1973, xii + 326 pp., $17.50.

Elliptic functions have attracted and fascinated many generations of mathematicians. For more than 150 years, these functions have kept their place in the center of mathematical interest and activity. Their appeal to us can perhaps be explained by their structural universality, what the author calls the intermingling of Analysis, Algebra and Arithmetic (the 3 Gaussian A's)– and thus a sizeable portion of Mathematics. In view of this, the theory of elliptic functions is considered to be a "deep" theory. Moreover, elliptic functions are the first nontrivial examples of the more general abelian functions. Not only do general theorems about abelian functions become explicit and more lucid in the case of elliptic functions, but also do special results about elliptic functions often constitute the first stepping stone on the way to their generalization in the abelian case.

Treatises on elliptic functions are numerous and it seems futile to attempt a classification. As to the book under review, its place in the existing literature is perhaps best described by saying that it continues the tradition of the classics by Weber and Fricke, including also more recent results which are connected with the names of Hasse, Deuring and Shimura. Specifically, a large part of the book is devoted to the body of results known under the name of "complex multiplication". These results are concerned with the so-called singular values of elliptic modular functions, and it is shown that they serve to generate abelian extensions of imaginary quadratic number fields. In addition to complex multiplication, the author also considers the case of nonsingular values of modular functions and the fields they generate. This includes the "generic" case, where the modular invariant j is transcendental, as well as the algebraic case where the value of j is algebraic but not integral. In these cases, the generated fields admit nonabelian automorphism groups; in fact those automorphism groups are closely related to the groups GL_2 (resp. SL_2) of 2×2 matrices, due to Shimura and Serre.

Already from these brief remarks it is evident that this book is leading straight into the center of contemporary research activity. During the past years, those topics have been enriched by new and striking results which generated a growing and widespread interest. This trend is reflected in the emphasis and the flavor of this book, which is to be regarded as an excellent introduction into the classical as well as the more recent literature. It is without doubt that this book, written by one of the prominent contemporary mathematicians, will exert a strong influence upon the present generation of mathematicians and the direction of their further research. Due to the author's vivid and unconventional style, the book will attract and inspire many scholars of the subject. Its basis was provided by one of the author's courses, and in its exposition it still enjoys the informality of lecture notes. Its timeliness seems to be more important than its form, which sometimes looks unfinished or tentative. In some instances, proofs are merely outlined or even wholly suppressed. The reviewer has found it advisable to consult Shimura's *Arithmetic theory of automorphic forms* as an excellent supplement. Compared to Shimura's book, the author points out that he himself has placed a somewhat different emphasis. First, he assumes less of the reader (he says) and starts the theory of elliptic functions "from scratch". He does not discuss Hecke operators, but includes several topics not covered by Shimura, among them e.g. the Kronecker limit formula and the discussion of values of modular functions constructed as quotients of theta functions.

Let us now describe the contents of the book in some more detail.

Part One contains the foundations of the general analytic theory of elliptic functions. These are defined as lattice-periodic meromorphic functions, and the elementary proofs of their main properties are indicated. Only such notions and results are included which are to be used in the arithmetical applications of the following parts. Elliptic functions are motivated such as to coordinatize elliptic curves; for the latter, the reader is assumed to have some knowledge of algebraic geometry in order to follow the author's remarks. The general policy is to tell the reader what is true in arbitrary characteristic, giving proofs mostly in characteristic zero only, using the transcendental parametrizations. Part one also contains an extended investigation of modular functions of higher level (in the sense of Klein and Fricke). In particular, the field of modular functions F is defined and studied not only over the complex numbers but also over the rational number field Q as base field. F is generated over Q by the modular invariant j, together with the so-called Weber (resp. Fricke) functions which are essentially the values of the Weierstrass $\wp$-function at the division points, up to certain normalizing factors. The main theorem about F over Q is the description of its group of automorphisms, which provides the basis for the reciprocity law of complex multiplication. According to Shimura, the automorphism group of $F|Q$ is isomorphic to a factor group of the rational adelic group $\mathrm{GL}_2(A)$, i.e. nonsingular 2×2 matrices over the rational adele ring $A = A(Q)$. The kernel of that factor group is the multiplicative group of Q, which is regarded as naturally imbedded into $\mathrm{GL}_2(A)$.

Part Two contains the theory of complex multiplication for elliptic curves with singular invariants. General class field theory is assumed, as well as the reduction theory of elliptic curves in the case of "good" reduction. The fundamental result is the famous congruence relation of Kronecker-Hasse for the singular values of the modular j-function. The proof is given first via Deuring's approach, i.e. by means of reduction theory and comparing the corresponding reduced curves. Later, the analytic proof of Hasse and Deuring is also presented. The basic theorems of complex multiplication are derived using ordinary ideals, and referring to class field theory given by ideal class groups. The relation to class field theory in the modern form, i.e. through the ideles, is also exhibited. But the central theme of part two (and in fact of almost the whole book) is the discussion of Shimura's general and beautiful reciprocity law. This theorem connects the maximal abelian Galois group over an imaginary quadratic number field k with a subgroup of the automorphism group of the modular function field $F|K$, as described above by means of $\mathrm{GL}_2(A)$. More precisely, let z be a generator of k, and assume $\mathrm{Im}(z) > 0$. Consider those modular functions $f \in F$ which are defined at z. Their (singular) values $f(z)$ generate the maximal abelian extension k_{ab} of k. The Galois group of k_{ab} over k is represented by the idele group of k (modulo principal ideles), by means of Artin's reciprocity law. If s is an idele of k, then let (s, k) denote the corresponding Artin symbol, which is an automorphism of k_{ab} over k. Shimura's reciprocity law describes the action of (s^{-1}, k) on the singular values $f(z)$. Namely, there is a certain automorphism σ of the modular function field F such that

$$f(z)^{(s^{-1},k)} = f^{\sigma}(z)$$

for every f. As mentioned above, σ can be represented by some 2×2 matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{GL}_2(A)$; this matrix is given by

$$s \cdot \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix}$$

where a, b, c, d are rational adeles. The proof given by the author contains substantial simplifications of Shimura's original. Part two also contains a chapter on Deuring's p-adic representations and its relation to isogenies. Also, Deuring's results about reduction and lifting singular curves (modulo p) are treated, and the same ideas are applied to study the reduction of modular functions ("Ihara's theory").

Part Three is devoted to elliptic curves with nonintegral invariants. These are studied by means of the Tate parametrization, which is a nonarchimedean counterpart of the ordinary parametrization by means of q-expansions. The theory of Tate is discussed at such length so as to include all that is needed to prove the isogeny theorem. This theorem asserts that two elliptic curves are isogenous if their associated p-adic Galois representations are isomorphic, for some prime p, provided the curves are defined over an algebraic number field and have non-p-integral invariants. The proof is due to Serre, and it is supplemented by the author to include also the case where the curves are defined over a function field (of characteristic zero), and their invariants are transcendental. Finally, the field of division points of an elliptic curve A over an algebraic number field K is studied. Let A_0 denote the group of division points of A (= points of finite order), and $K(A_0)$ the field generated by their coordinates over K. If A has no complex multiplication, it is known from Serre's work that the Galois group of $K(A_0)$ over K is an open subgroup of the product $\prod_p \mathrm{GL}_2(Z_p)$ (taken over all primes p). This important theorem is proved here under the additional assumption that the invariant of A is nonintegral.

Part Four enters into the multiplicative theory of elliptic (theta) functions, and the connection to L-series. After first dealing with the analytic theory, exhibiting the classical multiplicative functions and their formulas, the author defines the so-called Siegel functions, which are certain integral modular functions. Their singular values lie in certain well-defined ray class fields; their behavior under Galois automorphisms is deduced from the general reciprocity law of Shimura mentioned above. Two Kronecker limit formulas involving (multiplicative) elliptic functions are established, as well as their relation to L-series over imaginary quadratic number fields. In particular, their value at $s = 1$ is worked out, which plays an important role in algebraic number theory in connection e.g. with class number formulae.

The attentive reader who has travelled up to this point over the ocean of elliptic functions, with this book as his vessel and the author as his guide, will certainly be able to sail further on his own to the scenes of great discoveries past and present. The tour is to be highly recommended even though the sea sometimes may be going rough.

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Maximal orders, by I. Reiner, Academic Press, London, New York, San Francisco, 1975, xii + 395 pp., \$36.50.

The theory of orders is a fascinating and difficult subject which occupies much of the common ground between algebra and number theory. Since this theory is known to relatively few contemporary mathematicians, I will give a more than usually thorough survey of the general area, before discussing the book itself.

To facilitate the discussion, we start with the definition and a few examples. Let R be a Dedekind domain (that is, R is a Noetherian integral domain in which all nonzero prime ideals are maximal, and which is integrally closed in its field of fractions K. The last condition means that any element of K which is a zero of a monic polynomial with coefficients in R belongs to R. As examples, one may take principal ideal domains and the rings of algebraic integers of algebraic number fields). An R-order is intended to be a certain type of R-algebra. Some of the examples which should be included are the ring of all $n \times n$ matrices over R (for any $n \geqslant 1$), the group ring RG of a finite