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THE PRINCIPAL SYMBOL OF A DISTRIBUTION

BY ALAN WEINSTEIN

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In Hörmander’s theory of Fourier integral operators [1], a principal symbol is constructed for a certain class of distributions in such a way that, when the construction is applied to the Schwartz kernel of a pseudodifferential operator, one obtains the usual principal symbol of the operator. In this note, we describe a generalization of Hörmander’s construction which may be applied to an arbitrary distribution on a manifold. Details will appear in [4].

1. Local definition and invariance properties. For a complex vector space $V$, we define $V$-valued distributions on $\mathbb{R}^n$ by taking as test functions objects of the form $u = u(x)\,dx$, where $u(x)$ is a compactly supported $C^\infty$ function with values in $V^*$, and $dx$ is the density $|dx_1 \wedge \cdots \wedge dx_n|$. For $\tau > 0$, we define $u_\tau$ to be $u(\tau x)\,dx$. If $g$ is a $V$-valued distribution, and $\varphi$ is a $C^\infty$ function with $\varphi(0) = 0$, we define the family $\{g_\varphi^\tau\}_{\tau > 0}$ of distributions by

$$
\langle g_\varphi^\tau, u \rangle = \langle g, e^{-i\tau \varphi}u_\tau \rangle.
$$

For $N \in \mathbb{R}$, we write $g_\varphi^\tau \in O(\tau^N)$ if $\tau^{-N}g_\varphi^\tau$ remains bounded in distribution space [3] as $\tau \to \infty$.

**Lemma.** For every $g$ and $\varphi$, $g_\varphi^\tau \in O(\tau^N)$ for some $N \in \mathbb{R}$.

**Definition.** $\inf \{N | g_\varphi^\tau \in O(\tau^N) \} \in [-\infty, \infty)$ is called the order of $g$ at $\varphi$ and denoted by $O_\varphi(g)$.

**Theorem 1.** (a) If $O_\varphi(g) \leq N$ and $\psi(x) = \varphi(x) + \sum_{j,k} a_{jk} x_j x_k + O(x^3)$, then $g_\varphi^\tau - e^{-i\sum_{j,k} a_{jk} x_j x_k}g_\varphi^\tau \in O(\tau^{N-1/2})$.

(b) If $O_\varphi(g) \leq N$ and $A$ is a $C^\infty$ function with values in $\text{Hom}(V, V)$, then $A(g_\varphi^\tau) - A(0)g_\varphi^\tau \in O(\tau^{N-1/2})$.

(c) If $O_\varphi(g) \leq N$ and $\theta : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism with $\theta(0) = 0$, then $(\theta \circ g)_\varphi^\tau - (T_0 \circ \theta) \circ g_\varphi^\tau \in O(\tau^{N-1/2})$.

**Definition.** If $O_\varphi(g) \leq N$, the class of $\tau^{-N}g_\varphi^\tau$ modulo $O(\tau^{-1/2})$ is called the principal symbol of order $N$ for $g$ at $\varphi$.


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2 See final note added in proof.
2. **Global theory.** The space \( \mathcal{D}'(X; E) \) of distributions on a manifold \( X \) with values in a vector bundle \( E \) is defined as the dual of the space of compactly supported \( C^\infty \) sections of \( E^* \otimes \Omega_X^1 \). (\( \Omega_X^1 \) = densities on \( X \).) Using local coordinates on \( X \) and local trivializations of \( E \), and applying Theorem 1, we can define for each \( (x, \xi) \in T^*X \) the order \( O_{(x, \xi)}(g) \) of \( g \) at \( (x, \xi) \).

To define the principal symbol in an invariant way, we must take into account the effect of changes in \( \varphi \) and coordinate changes as given by Theorem 1. For each \( (x, \xi) \in T^*X \), the 2-jets of functions \( \varphi \in C^\infty(X; \mathbb{R}) \) with \( \varphi(x) = 0 \) and \( d\varphi(x) = \xi \) can be identified with the elements of the space \( L_{(x, \xi)} \) of lagrangian subspaces in \( T^*_X \). The additive group \( Q_x \) of homogeneous quadratic functions on \( T_X \) acts simply and transitively on \( \mathcal{L}(x, \xi) \), and it also acts on \( \mathcal{D}'(T_X; E_x) \) by \( (a, g) \mapsto e^{-i\alpha}g \). The space \( \mathcal{U}^N_{(x, \xi)}(X; E) \) is defined to consist of the \( Q_x \)-equivariant maps from \( \mathcal{L}^N_{(x, \xi)} \) to \( \mathcal{D}'(T_X; E_x) \), and \( S^N_{(x, \xi)}(X; E) \) is defined as the space of families \( \{g_\tau\}_{\tau > 0} \in \mathcal{U}^N_{(x, \xi)}(X; E) \) with \( g_\tau \in \mathcal{O}(\tau^N) \).

Now, if \( O_{(x, \xi)}(g) \leq N \), Theorem 1 implies that the principal symbol \( \sigma^N_{(x, \xi)}(g) \) of order \( N \) for \( g \) at \( (x, \xi) \) is well defined as an element of \( S^N_{(x, \xi)}(X; E) \). If \( \sigma^N_{(x, \xi)}(g) \) is of the form \( g_0 + S^{-1/2}_{(x, \xi)}(X; E) \), where \( g_0 \) is a constant in \( \mathcal{U}_{(x, \xi)}(X; E) \), we say that \( g \) is homogeneous of order \( N \) at \( (x, \xi) \) and, by abuse of notation, write \( \sigma^N_{(x, \xi)}(g) = g_0 \). If \( O_{(x, \xi)}(g) = N \), we simply call \( \sigma^N_{(x, \xi)}(g) \) the principal symbol of \( g \) at \( (x, \xi) \) and denote it by \( \sigma_{(x, \xi)}(g) \).

As a first step toward a general calculus of principal symbols, we have:

**THEOREM 2.** Let \( P \) be a pseudodifferential operator of order \( k \) and type \((1, 0)\) from \( E \) to \( F \) with homogeneous principal symbol \( (x, \xi) \mapsto p(x, \xi) \in \text{Hom}(E_x, F_x) \). If \( O_{(x, \xi)}(g) \leq N \), \( \xi \neq 0 \), then \( O_{(x, \xi)}(Pg) \leq N + k \), and \( \sigma^{N+k}_{(x, \xi)}(Pg) = p(x, \xi)\sigma^N_{(x, \xi)}(g) \).

**COROLLARY.** \((x, \xi) \notin WF(g) \Rightarrow O_{(x, \xi)}(g) = -\infty.\)

3. **The principal symbol of a Fourier integral distribution.**

**DEFINITION.** If \( S \) is a subspace of \( T_X \), an \( E_x \)-valued \( \delta \)-function on \( S \) is a distribution \( \delta \in \mathcal{D}'(T_X; E_x) \) such that:

(a) \( \delta \) is continuous for the \( C^0 \) topology;
(b) \( \delta \) is supported on \( S \);
(c) \( \delta \) is translation-invariant by \( S \).

For example, the \( \delta \)-densities on \( S \) correspond to the translation invariant measures on \( S \) and form a 1-dimensional space.

Now let \( K \subseteq T_{(x, \xi)}T^*X \) be a lagrangian subspace, with projection \( \overline{K} \) in \( T_X \).

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3 This construction, as well as that of \( \eta_{(x, \xi)}(X; E) \) itself, is closely related to the symplectic spinors of Kostant and Sternberg [2].
For each $E_x$-valued $\delta$-function $\delta$ on $K$ there is a unique $\delta_K \in \mathcal{U}_{(x,\xi)}(X; E)$ which assigns $\delta$ to every $L \in L_{(x,\xi)}$ such that $\dim(L \cap K) = \dim K$. The set of all $\delta_K$, as $\delta$ runs over the $E_x$-valued $\delta$-functions on $K$, is a $\dim E_x$ dimensional subspace $\Delta_K(E)$ of $\mathcal{U}_{(x,\xi)}(X; E)$.\(^3\)

**Theorem 3.** Let $g \in \mathcal{V}'(X; E)$ be a Fourier integral distribution associated with the conic lagrangian submanifold $\Lambda \subset T^*X$ and having order $m$, type $(1, 0)$ and homogeneous principal symbol. Let $(x, \xi) \in \Lambda$, $K = T_{(x,\xi)}\Lambda$.

(a) $g$ is homogeneous of order $\bar{m} = m + \frac{1}{2} \dim X$ at $(x, \xi)$.

(b) $\sigma_{(x,\xi)}^\bar{m}(g) \in \Delta_K(E)$.

(c) If $E$ is the bundle of $\frac{1}{2}$ densities on $X$, then $\Delta_K(E)$ is naturally isomorphic with the fibre over $(x, \xi)$ of the symbol bundle $\Omega_{1/2} \otimes L$ of [1], and $\sigma_{(x,\xi)}^\bar{m}(g)$ is equal to the principal symbol as given in [1].

**Remark.** We can show directly that $\Delta_K(E)$ depends smoothly on lagrangian $K \subset T_{(x,\xi)}(T^*X)$, thus giving a new, analytic, construction of the symbol bundle.

**Added in Proof (June 2, 1976).** As A. Douady has pointed out to me, the set \( \{ N \mid g^N \in \mathcal{O}(\tau^N) \} \) could be an interval of the form $(a, \infty)$. In this case, we should define $\mathcal{O}(\tau^N)$ to be $a^+$, where $a^+$ lies by convention between $a$ and any number greater than $a$.

**References**


**Institut Des Hautes Études Scientifiques, Bures-sur-Yvette, France**

Current address: Department of Mathematics, University of California, Berkeley, California 94720