DEGENERATION AND SPECIALIZATION IN ALGEBRAIC FAMILIES OF VECTOR BUNDLES

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Introduction. If $X$ is a nonsingular $d$-dimensional projective variety (over an algebraically closed field, $k$), then by a vector bundle on $X$ we understand a torsion-free, coherent $\mathcal{O}_X$-module. The classical notion of vector bundle (= locally free $\mathcal{O}_X$-module) is subsumed in the new definition and agrees with it when $X$ is a curve. The new notion is necessary for the study of moduli problems [G], [L] and because the classical definition is too rigid for higher dimensional varieties [L]. However, given a vector bundle on $X$, there exists an open subset $U$ of $X$ containing all points of codimension $\leq 1$ such that the vector bundle is locally free over $U$. Chern classes can always be defined for such bundles.

We fix, once and for all, a very ample divisor class, $H$, on $X$. Then for any prescheme $S$ over $k$, the line bundle $p^*H$ on $X \times_k S$ (where $p: X \times_k S \to X$ is the projection) is very ample with respect to $S$. As usual, set $F(n) = F \otimes H^\otimes n$, then the Hilbert polynomial $p_F(n)$ is defined and, by the Riemann-Roch Theorem, $p_F(n)$ has an expression in terms of Chern classes. Indeed, if we set

$$\deg F = (c_1(F) \cdot H^{d-1}) \quad \text{(intersection no.)},$$

and refer to $\deg F$ as the degree of $F$ (more properly, the $H$-degree of $F$), then

$$\frac{p_F(n)}{\rk F} = \delta(X) \frac{n^d}{d!} + \left( \frac{\deg F}{\rk F} - \frac{\deg K}{2} \right) \frac{n^{d-1}}{(d-1)!} + \text{terms of lower degree in } n.$$

The quantities $\widetilde{\mu}(F) = p_F(n)/\rk F$, $\mu(F) = (\deg F)/\rk F$ are fundamental for what we shall do. The former was introduced by Gieseker, the latter by Take-moto [T], and both were inspired by results of Mumford for curves. We shall concentrate on $\mu$; however, everything we do carries over to $\widetilde{\mu}$ suitably interpreted. Call $\mu(F)$ the slope of $F$.

The bundle $F$ is semistable (resp. stable) if for every proper subbundle, $G$, $\mu(G) \leq \mu(F)$ (resp. $\mu(G) < \mu(F)$). Our bundle $F$ is unstable if it is not semistable. We regard an unstable bundle as "more degenerate" than a semistable bundle, etc.

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Main results. Let $F$ be a bundle on $X$, as above. A *Harder-Narasimhan Flag for $F$* (abbreviated $\text{HNF}(F)$) is a descending chain of subbundles of $F$,

$$(*) \quad F > F_{t-1} > \cdots > F_1 > (0),$$

having the properties:

(a) each factor bundle $F_j/F_{j-1}$ is semistable;

(b) $\mu(F_{j+1}/F_j) < \mu(F_j/F_{j-1})$, $1 \leq j \leq t - 1$.

(These chains were introduced in [HN] in the case $X$ was a curve, for other purposes.)

**Theorem 1.** *Every bundle, $F$, on the $d$-dimensional, nonsingular, irreducible, projective variety, $X$, possesses a unique HNF. Any two flags ($*$) satisfying (a) and (b) are identical. Moreover, if $F$ is itself semistable, then $F$ possesses a flag ($*$) in which

(a) each factor bundle is stable, and

(b) $\mu(F_{j+1}/F_j) = \mu(F_j/F_{j-1})$."

Theorem 1 is not difficult to prove if one makes full use of the new definition of vector bundle—it is false otherwise. Given a bundle $F$, if we plot in the (rk, deg)-plane the points whose coordinates are the ranks and degrees of the bundles occurring in ($*$) for $F$, we obtain a polygon which we call the *Harder-Narasimhan Polygon for $F$* ($\text{HNP}(F)$). The slopes of the sides of this polygon are exactly the numbers $\mu(F_1)$, $\mu(F_2/F_1)$, etc. occurring in (b) above. Theorem 1 states that every $F$ possesses a unique HNP, and that $\text{HNP}(F)$ is a convex polygon.

Let $X$ be a nonsingular, irreducible, projective $k$-variety and let $S$ be a scheme over $k$. A vector bundle on $X \times_k S$, flat over $S$, will be called an *algebraic family of vector bundles on $X$ parametrized by $S$*. If $F$ is a family on $X$ and $s \in S$, we let $F_s$ denote the pull-back of $F$ to the fibre, $X_s$, of $X \times_k S$ over $s$. The divisor class $H$ on $X$ induces divisor classes $p^*H$ and $H_s$ on $X \times_k S$ and $X_s$, respectively, by pull-back. These are very ample as $H$ is, and semistability is measured via these given very ample sheaves.

**Theorem 2.** *Given $X$, $S$, and $F$ on $X \times_k S$, as above, form $\text{HNP}(F_s)$ for each $s \in S$. If $t_0 \in S$ is a specialization of $s \in S$, then $\text{HNP}(F_{t_0})$ lies on or above $\text{HNP}(F_s)$. That is, the Harder-Narasimhan polygon rises under specialization.*

When the moduli scheme for stable bundles exists (i.e., for $X$ a curve [M] or a surface [G]), one can show that $\text{HNP}(F_s)$ is a constructible function of $s$, and therefore Theorem 2 implies it is upper semicontinuous. Using the upper semi-

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2 I want to thank D. G. Quillen for suggesting that my earlier work for curves, $X$, be formulated as in Theorem 2—thereby inducing me to prove Theorem 2 in general.
continuity, we have obtained a fundamental map from vector bundles on $X \times_k S$ to collections of $r - 1$ algebraic cycles (with nonnegative coefficients) of $S$, $r$ being the rank of the bundles.

In the simplest case, let $Y$ be a ruled surface with base curve $C$, and let $Y_c$ denote the generic fibre. Fix a convex polygon, $P$, with vertices at $(0, 0), (1, d_1), \ldots, (r, d_r)$ (the $d$'s are given integers), and let $\text{Vect}(Y; P)$ denote the set map of rank $r$ bundles on $Y$ whose HNP at the generic fibre $Y_c$ is the given polygon $P$. Then we obtain a

$$\text{Vect}(Y; P) \xrightarrow{\theta} \prod_{r-1 \text{ factors}} \text{Hilb}(C).$$

It turns out that two bundles on $Y$ have the same image under $\theta$ if and only if their pull-backs to each fibre of $Y$ over $C$ are isomorphic, moreover the map $\theta$ is surjective. These matters are discussed in $[S_1], [S_2], [S_3]$.

REFERENCES


[S_2] ———, Vector bundles on ruled surfaces (preprint).


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