1. Introduction. In the book, *Theory of games and economic behavior* (1944), J. von Neumann and O. Morgenstern introduced a theory of solutions (or stable sets) for multi-person cooperative games in characteristic function form. A longstanding conjecture has been that the union of all solutions of any particular game is a connected set. (E.g., see [3].) This announcement describes a twelve-person game for which the conjecture fails. The essential definitions for an n-person game will be reviewed briefly before the counterexample is presented. A sketch of the proof is presented here, and the details will appear elsewhere.

2. The model. An n-person game is a pair \((N, v)\) where \(N = \{1, 2, \ldots, n\}\) is the set of players and \(v\) is a characteristic function on \(2^N\), i.e., \(v\) assigns the real number \(v(S)\) to each subset \(S\) of \(N\) and \(v(\emptyset) = 0\). The set of imputations is

\[ A = \{ x: \sum_{i \in N} x_i = v(N) \text{ and } x_i \geq v(\{i\}) \text{ for all } i \in N \} \]

where \(x = (x_1, x_2, \ldots, x_n)\) is a vector with real components. For any \(S \subset N\), let \(x(S) = \sum_{i \in S} x_i\). For any \(X \subset A\) and nonempty \(S \subset N\), define \(\text{Dom}_S X\) to be the set of all \(x \in A\) such that there exists a \(y \in X\) with \(y_i > x_i\) for all \(i \in S\) and with \(y(S) \leq v(S)\). Let \(\text{Dom} X = \bigcup_{\emptyset \neq S \subset N} \text{Dom}_S X\). A subset \(V\) of \(A\) is a solution if \(V \cap \text{Dom} V = \emptyset\) and \(V \cup \text{Dom} V = A\). The core of a game is

\[ C = \{ x \in A: x(S) \geq v(S) \text{ for all nonempty } S \subset N \}. \]

For any solution \(V\), \(C \subset V\) and \(V \cap \text{Dom} C = \emptyset\).

A characteristic function \(v\) is superadditive if \(v(S \cup T) \geq v(S) + v(T)\) whenever \(S \cap T = \emptyset\). The game below does not have a superadditive \(v\) as is assumed in the classical theory, but it is equivalent solutionwise to a game with a superadditive \(v\). (See [1, p. 68].)

3. Example. The 13 vital coalitions for our example consist of \(N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}\) and elements from three classes:

\[ B = \{ \{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}, \{9, 10\}, \{11, 12\} \}. \]
\[ S = \{\{1, 3, 6, 7, 9, 11\}, \{1, 4, 5, 7, 9, 11\}, \{2, 3, 5, 7, 9, 11\}\}, \]
\[ T = \{\{1, 3, 8\}, \{1, 5, 10\}, \{3, 5, 12\}\}. \]

And \( v \) is given by:
\[ v(N) = 6, \]
\[ v(S) = 1 \text{ for all } S \in G \text{ B}, \]
\[ v(S) = 4 \text{ for all } S \in G \text{ S}, \]
\[ u(S) = 1 \text{ for all } S \in G \text{ T}, \text{ and } u(S) = 0 \text{ for all other } S \in C \text{ N}. \]

For this game \( A = \{x: x(N) = 6 \text{ and } x_i \geq 0 \text{ for all } i \in N\} \). Consider also the six-dimensional hypercube
\[ B = \{x \in A: x(S) = 1 \text{ for all } S \in B\}. \]

The core \( C \) is the intersection of \( C(S) \) and \( C(T) \) where
\[ C(S) = \{x \in B: x(S) \geq 4 \text{ for all } S \in S\}, \]
\[ C(T) = \{x \in B: x(S) \geq 1 \text{ for all } S \in T\}. \]

\( C \) is a proper superset of the convex hull of the six vertices of \( B \) which have \( x_i = 1 \) for \( i \) equal to five of the six odd indices 1, 3, 5, 7, 9 and 11, and \( x_{i+1} = 1 \) when \( i \) is the remaining odd numbered player. Let \( \text{Dom}_G X = \bigcup_{S \in B} \text{Dom}_S X \).

Note that \( \text{Dom}_G C \supset A - B \), and hence any solution \( V \) for our game is a subset of \( B \).

4. Outline of proof. First, note that any component of an \( x \in B \) has a maximum value of \( x_i = 1 \). Consequently, the following three sets are contained in any solution \( V \), i.e., they are subsets of \( \bigcap V \):
\[ E = \{x \in B: x_i = x_j = 1 \text{ for } i \neq j \text{ and } \{i, j\} \subset \{1, 3, 5\}\}, \]
\[ F = \{x \in C(T): x_p = 1 \text{ for } p = 7, 9 \text{ or } 11\}, \]
\[ P = \{(0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0)\}. \]

Next, we can show that \( \bigcup V \) must be a disconnected set. Let \( G = \{x \in B: x(\{7, 9, 11\}) \leq 1\}, \)
\[ G^0 = \{x \in B: x(\{7, 9, 11\}) < 1\}, \text{ and } P' = \{x \in G: x_2 = x_4 = x_6 = 1\}. \]

Throughout this section the indices \( i, j \) and \( k \) represent some ordering of the distinct indices 1, 3 and 5. The subset \( H \) of \( E \) consisting of the three triangular regions
\[ H_i = \{x \in G: x_{i+1} = x_j = x_k = 1; x_7 + x_9 + x_{11} = 1\} \]
is in \( \bigcap V \) and \( \text{Dom}_S N \supset G^0 - (E \cup P') \). The subset \( J \) of \( F \) consisting of the three triangular regions
\[ J_1 = \{x \in F: x_1 = x_7 = x_9 = 1, x_3 + x_5 + x_{12} = 1\}, \]
\[ J_3 = \{x \in F: x_3 = x_7 = x_{11} = 1, x_1 + x_5 + x_{10} = 1\}, \]
\[ J_5 = \{x \in F: x_5 = x_9 = x_{11} = 1, x_1 + x_3 + x_8 = 1\} \]
is also in \( \bigcap V \) and \( \text{Dom}_S J \supset B - C(T) \supset P' - P \). So any \( x \in \bigcup V - P \) either has \( x \in E \) or \( x \in B - G^0 \), i.e., \( x_i = x_j = 1 \) or \( x(\{7, 9, 11\}) \geq 1 \). Such \( x \) are clearly disconnected from the singleton \( P \subset \bigcap V \).

Finally, it is necessary to demonstrate that this game does possess at least one solution. \( V' = C \cup E \cup F \cup P \) is in any solution \( V \), and \( V' \) can be enlarged to a solution in two steps. First, include the set of imputations \( L \) in \( C(T) - \)
which is simultaneously maximal with respect to all three of the relations “$\text{Dom}_S$” for $S \in S$. Clearly $L \subseteq \bigcap V$. Next, pick a particular $S^i = \{i + 1, i, k, 7, 9, 11\} \in S$ and then add in those elements $L^i$ in $C(T) - (V' \cup L \cup \text{Dom}(V' \cup L))$ which are maximal with respect to the relation “$\text{Dom}_S^i$” and are at the same time symmetrical in the sense that $x_j = x_k$. It requires some detail to describe the sets $L$ and $L^i$ explicitly, and to verify that the resulting sets $V^i = V' \cup L \cup L^i$ are solutions for our example. These will appear elsewhere.

5. **Remarks.** At one time it was apparently believed that proving the union of all solutions connected could be a major step in showing that every game has a solution. It is now known [2] that a solution need not exist for every game. On the other hand, it is possible that results on disconnecting $V'$ might be useful in the resolution of important open questions about whether solutions always exist for games with full-dimensional cores, with empty cores, or which are constant-sum.

**REFERENCES**


SCHOOL OF OPERATIONS RESEARCH AND CENTER FOR APPLIED MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NEW YORK 14853