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STABILITY AND GROWTH ESTIMATES FOR VOLTERRA INTEGRODIFFERENTIAL EQUATIONS IN HILBERT SPACE

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Let H, H_+ be real Hilbert spaces with H_+ dense in H and $H_+ \subset H$, algebraically and topologically; the inner products on H, H_+ are denoted by $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle_+$, respectively. As in [1], let H_- denote the dual of H_+ via the inner product of H so that H_- is the completion of H under the norm

$$\|\mathbf{w}\|_- = \sup_{v \in H_+} \frac{|\langle v, w\rangle|}{\|v\|_+}.$$

By $L(H_+, H_-)$ we denote the space of bounded linear operators from H_+ to H_-. For $0 \leqslant t < T$, $T > 0$ an arbitrary real number, we consider the initial-value problem

$$\mathbf{u}_{tt} - \mathbf{N}\mathbf{u} + \int_{-\infty}^{t} \mathbf{K}(t-\tau)\mathbf{u}(\tau)\,d\tau = \mathbf{0}, \tag{1}$$

$$\mathbf{u}(0) = \mathbf{f}, \qquad \mathbf{u}_t(0) = \mathbf{g}, \tag{2}$$

where $\mathbf{N} \in L(H_+, H_-)$ is symmetric and $\mathbf{K}(t), \mathbf{K}_t(t) \in L^2((-\infty, \infty); L(H_+, H_-))$. We also assume that

$$\mathbf{u}(\tau) = \mathbf{U}(\tau), \qquad -\infty < \tau < 0, \tag{3}$$

where $\mathbf{U}(t) \in C^1((-\infty, 0); H_+)$ is prescribed and satisfies $\lim_{t\to 0^-} \mathbf{U}(t) = \mathbf{f}$, $\lim_{t\to 0^-} \mathbf{U}_t(t) = \mathbf{g}$, $\lim_{t\to -\infty} \|\mathbf{U}(t)\|_+ = 0$ and $\int_{-\infty}^0 \|U(t)\|_+\,dt < \infty$.

In [2] we have proved the following basic result concerning solutions $\mathbf{u} \in C^2([0, T); H_+)$ for which $\mathbf{u}_t \in C^1([0, T); H_+)$ and $\mathbf{u}_{tt} \in C([0, T); H_-)$. Let

$$\mathcal{N} = \left\{ \mathbf{w} \in C^2([0, T); H_+) \,\middle|\, \sup_{[0,T)} \|\mathbf{w}(t)\|_+ \leqslant N^2 \right\}$$

for some real number N. Then we have

THEOREM (BLOOM [2]). *Let* $\mathbf{u} \in \mathcal{N}$ *be any solution of* (1)–(3) *and define*

$$F(t; \beta, t_0) = \|\mathbf{u}(t)\|^2 + \beta(t + t_0)^2, \qquad 0 \leqslant t < T,$$

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where β, t_0 *are arbitrary nonnegative real numbers. Then provided* $\mathbf{K}(t)$ *satisfies*

$$-\langle v, \mathbf{K}(0)v\rangle \geqslant \kappa \|v\|_+^2, \forall v \in H_+ \tag{5a}$$

with

$$\kappa \geqslant T\gamma \sup_{[0,\infty)} \|\mathbf{K}_t(t)\|, \tag{5b}$$

[1]

$F(t;\beta, t_0)$ *satisfies*

$$FF'' - F'^2 \geqslant -2F(2G(0) + \beta), \qquad 0 \leqslant t < T, \tag{6}$$

where

$$G(t) = E(t) + k_1 \sup_{[0,\infty)} \|\mathbf{K}(t)\| + k_2 \sup_{[0,\infty)} \|\mathbf{K}(t)\| \tag{7}$$

with $E(t) \equiv \frac{1}{2}\langle \mathbf{u}_t(t), \mathbf{u}_t(t)\rangle - \frac{1}{2}\langle \mathbf{u}(t), \mathbf{N}\mathbf{u}(t)\rangle$, *the total energy and*

$$k_1 = \frac{1}{2}\gamma\left[TN^4 + (N^2 + 2\|\mathbf{f}\|_+)\int_{-\infty}^{0} \|\mathbf{U}(\tau)\|_+ \, d\tau\right],$$

$$k_2 = TN^2\gamma \int_{-\infty}^{0} \|\mathbf{U}(\tau)\|_+ \, d\tau.$$

The proof of this theorem proceeds via a logarithmic convexity argument due to Knops and Payne [2]. As no definiteness conditions are imposed on $\mathbf{N}$, the problem (1)–(3) is, in general, non-well-posed and the existence and uniqueness results of Dafermos [1] do not apply.

From (6) there follows a variety of results concerning the stability and growth behavior of solutions $\mathbf{u} \in \mathcal{N}$ to (1)–(3); proofs of the sample results given below, as well as several others, along with applications to initial-boundary value problems arising in the theory of isothermal linear viscoelasticity may be found in [2] and [4]; in a forthcoming work [5] our results will be applied so as to study the stability and growth behavior of electric displacement fields in non-conducting material dielectrics.

THEOREM I (BLOOM [2]). *Let* $\mathbf{u} \in \mathcal{N}$ *be any solution of* (1)–(3) *for which* $E(0) \leqslant -k$, *for some* $k > 0$. *If* $\mathbf{K}(t)$ *satisfies* (5a), (5b) *and*

$$\sup_{[0,\infty)} \|\mathbf{K}(t)\| \leqslant \gamma kT/(k_1 T\gamma + k_2), \tag{8}$$

then

$$\|\mathbf{u}(t)\|^2 \geqslant \|\mathbf{f}\|^2 \exp\{\langle 2f, g\rangle t/\|\mathbf{f}^2\|\}, \qquad 0 \leqslant t < T, \tag{9}$$

whenever $\langle \mathbf{f}, \mathbf{g}\rangle \geqslant 0$ *with* $\mathbf{f} \neq \mathbf{0}$.

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[1] γ is the embedding constant, i.e., as $H_+ \subset H$ topologically, $\|v\| \leqslant \gamma\|v\|_+, \forall v \in H_+$.

THEOREM II (BLOOM [2]). *Let* $\mathbf{u} \in N$ *be any solution of* (1)–(3) *and suppose that*

$$E(0) > -(k_1 T\gamma + k_2) \sup_{[0,\infty)} \|\mathbf{K}_t(t)\| \tag{10}$$

and

$$\langle \mathbf{f}, \mathbf{g} \rangle \, 2\sqrt{2+\epsilon}\,(G(0))^{1/2} \|\mathbf{f}\| \tag{11}$$

with $\epsilon = \frac{1}{4}\mu\langle \mathbf{f}, \mathbf{g} \rangle$, *for some* $\mu > 0$. *Then provided* $\mathbf{K}(t)$ *satisfies* (5a), (5b), $\mathbf{u}$ *satisfies*

$$\|\mathbf{u}(t)\|^2 + \mu^{-1} \geqslant (\|\mathbf{f}\|^2 + \mu^{-1})(t/\hat{t}_0 + 1)^\epsilon \exp\{\delta_\mu(\hat{t}_0; \epsilon)t\} \tag{12}$$

for $0 \leqslant t < T$, *where* $\hat{t}_0 \equiv \langle \mathbf{f}, \mathbf{g} \rangle / 4G(0)$ *and*

$$\delta_\mu(\hat{t}_0; \epsilon) \equiv 2\left\{\frac{\langle \mathbf{f}, \mathbf{g} \rangle + \hat{t}_0/\mu}{\|\mathbf{f}\|^2 + 1/\mu}\right\} - \frac{(2+\epsilon)}{\hat{t}_0}.$$

THEOREM III (BLOOM [4]). *Let* $\mathbf{u} \in N$ *be any solution of* (1)–(3) *where* $\mathbf{K}(t)$ *satisfies* (5a) *and* (5b). *If*

$$E(0) \leqslant -k \sup_{[0,\infty)} \|\mathbf{K}(t)\| \tag{13}$$

with $k \geqslant k_1 + \gamma k_2/T$, *then* $\mathbf{u}$ *satisfies*

$$\|\mathbf{u}(t)\|^2 \leqslant A[\max(\|\mathbf{f}\|^2, \|\mathbf{g}\|^2)]^{2(1-\delta)} \tag{14}$$

for $0 \leqslant t < T$, *where* $A > 0$ *and* $\delta = t/T$.

THEOREM IV (BLOOM [4]). *Let* $\mathbf{u} \in N$ *be any solution on* (1)–(3) *and suppose that* $E(0) > -\tilde{k}$, *for some* $\tilde{k} \geqslant 0$. *If* $\mathbf{K}(t)$ *satisfies* (5a), (5b) *and*

$$\text{(i)} \sup_{[0,\infty)} \|\mathbf{K}_t(t)\| > \tilde{k}/(k_1 T\gamma + k_2), \tag{16}$$

$$\text{(ii)} \lim_{T\to\infty} \frac{1}{T} \ln\{\|\mathbf{u}(T)\|^2 + \beta(T+t_0)^2\} = 0 \tag{17}$$

for β, t_0 *nonnegative constants satisfying* $\beta t_0^2 \leqslant \|\mathbf{f}\|^2$, *then* $\mathbf{u}$ *satisfies*

$$\|\mathbf{u}(t)\|^2 \leqslant \Phi(t_0, T; \epsilon)\|\mathbf{f}\|^2, \quad 0 \leqslant t < T, \tag{18}$$

where $\Phi(t_0, T; \epsilon) \equiv 2(T/t_0 + 1)^{2+\epsilon}$ *with* $\epsilon = G(0)/\beta$.

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