A GRADIENT ESTIMATE AT THE BOUNDARY
FOR SOLUTIONS OF QUASILINEAR ELLIPTIC EQUATIONS

BY RONALD GARIEPY AND WILLIAM P. ZIEMER

Communicated by Hans F. Weinberger, April 8, 1976

1. Introduction. The purpose of this note is to present a result concerning
regularity at the boundary of bounded, weak solutions of equations of the form
\[ \text{div } A(x, u, u_x) = B(x, u, u_x) \]
where \( A \) and \( B \) are, respectively, vector and scalar valued Baire functions defined
on \( \Omega \times R^1 \times R^n \) that satisfy
\[ |A(x, u, w)| \leq a_0 |w|^{p-1} + a_1 |u|^{p-1} + a_2, \]
\[ |B(x, u, w)| \leq b_0 |w|^p + b_1 |w|^{p-1} + b_2 |u|^{p-1} + b_3, \]
\[ w \cdot A(x, u, w) \geq c_0 |w|^p - c_1 |u|^p - c_2. \]
Here, \( \Omega \) is an open subset of \( R^n \), \( 1 < p < n \), \( c_0 > 0 \), \( a_0 \geq 0 \), \( b_0 \geq 0 \), and the
remaining coefficients are nonnegative, measurable functions in the respective
Lebesgue classes
\[ a_1, a_2 \in L_{n/p-1}(\Omega), \quad b_1 \in L_{n/1-\delta}(\Omega), \]
\[ c_1, c_2, b_2, b_3 \in L_{n/p-\delta}(\Omega) \quad \text{where} \quad 0 \leq \delta < 1. \]
A weak solution of (1) is a function \( u \) in the Sobolev space \( W^1_p(\Omega) \) that satisfies
\[ \int_{\Omega} A \cdot \nabla \varphi + B \cdot \varphi = 0 \]
whenever \( \varphi \) is a smooth function with compact support in \( \Omega \).

It has been shown in [LU], [S], and [T] that a weak solution of (1) is
Hölder continuous on compact subsets of \( \Omega \). In connection with boundary regulat-
arity, it was established in [LSW] that a point \( x_0 \in \partial \Omega \) is regular for solutions
of linear, uniformly elliptic equations in divergence form with bounded, measurable
coefficients if and only if \( x_0 \) is regular for Laplace’s equation. Later,
Stampacchia [ST] extended this result to a wider class of linear elliptic equations.

For solutions of quasilinear equations of the form \( \text{div } A(x, u_x) = 0 \), but subject
to conditions more restrictive than (2), Maz’ya [M] established regularity at \( x_0 \)
\( \in \partial \Omega \) provided the following condition is satisfied:
Here \( B(x_0, r) \) denotes the \( n \)-ball of radius \( r \) centered at \( x_0 \) and \( \Gamma_p \) is a capacity defined on all sets \( E \subset \mathbb{R}^n \)

\[
\Gamma_p(E) = \inf \left\{ \int |\nabla f|^p \right\}
\]

where the infimum is taken over all \( f \in L_{np/(n-p)} \cap W^1_p(\mathbb{R}^n) \) for which \( E \subset \text{int}\{x : f(x) \geq 1\} \). In view of the fact that \( \Gamma_2 \) is Newtonian capacity, one observes that (3) is precisely the classical Wiener condition when \( p = 2 \).

2. The main results. Given a continuous function \( f \) on \( \partial \Omega \) and \( x_0 \in \partial \Omega \), we will say \( u(x_0) \leq f(x_0) \) weakly for functions \( u \in W^1_p(\Omega) \) provided that whenever \( \eta \) is a smooth function supported in \( B(x_0, r) \) and \( f < k \) in \( B(x_0, r) \cap \partial \Omega \), then \( (u - k)^+ \eta \in W^1_{p,0}(\Omega) \). A similar definition is given for \( u(x_0) > f(x_0) \) weakly and, therefore, we can give meaning to \( u(x_0) = f(x_0) \) weakly.

As a direct consequence of the gradient estimate below, (5), we obtain the following

**Theorem.** Suppose \( f \) is a continuous function on \( \partial \Omega \) and let \( u \in W^1_p(\Omega) \) be a bounded weak solution of (1) such that \( u(x_0) = f(x_0) \) weakly. If (3) holds and

\[
\int_0^1 \frac{\|a_1 + a_2\|^{1/(p-1)}_{n/(p-1), B(x_0, r)}}{r} \, dr < \infty,
\]

then \( \lim_{x \to x_0 : x \in \Omega} u(x) = f(x_0) \).

The notation in (4) indicates the norm of \( a_1 + a_2 \) taken relative to the \( n \)-ball \( B(x_0, r) \). Of course, if it is assumed that \( a_1, a_2 \in L_q \) where \( q > n/(p - 1) \), then clearly (4) is satisfied.

It is interesting to observe that the regularity results of [LSW], [ST], and [M] are obtained by employing potential-theoretic techniques, whereas ours is based primarily on information obtained from the differential equation itself. Indeed, the following estimate is the vital component.

If \( u \in W^1_p(\Omega) \) is a bounded, weak solution of (1), let \( \mu(r) = \sup \{u(x) : x \in B(x_0, r)\} \), where \( x_0 \in \partial \Omega \). Suppose \( k > f(x_0) \) and let \( u_k = (u - k)^+ \).

**Theorem.** There is a constant \( C \) depending only on \( n, p \), the bound for \( u \), the coefficients in (2), and \( \delta \) such that for all sufficiently small \( r \),

\[
r^{p-n} \int_{B(x_0, r/4)} |\nabla u_k|^p \leq C [\mu(2r) - \mu(r) + a(r)]^{p-1}
\]

whenever \( u(x_0) \leq f(x_0) \) weakly and where
QUASILINEAR ELLIPTIC EQUATIONS

\[ a(r) = r + \|a_1 + a_2\|^{1/(p-1)}_{n/(p-1),B(x_0,r)} + \|c_1 + c_2 + b_2 + b_3\|^{1/(p-\delta/2)}_{n/(p-\delta/2),B(x_0,r)}. \]

Suppose \( u \) is a solution of (1) such that \( u(x_0) = f(x_0) \) weakly but that

\[ \lim_{x \to x_0; x \in \Omega} u(x) \neq f(x_0). \]

If (4) holds the gradient estimate (5) is used to show that there is a set \( E \) which is \( \Gamma_p \)-thin at \( x_0 \) (see [ME] for definition) such that \( u(x) \) tends to a limit as \( x \to x_0, \ x \notin E \). Thus, in terms analogous to the classical case, \( u \) has a \( \Gamma_p \)-fine limit at \( x_0 \).

Proofs of these and other results will appear elsewhere.

BIBLIOGRAPHY


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KENTUCKY, LEXINGTON, KENTUCKY 40506

DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, INDIANA 47401

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use