

ON THE TOPOLOGICAL STRUCTURE OF SIMPLY-CONNECTED ALGEBRAIC SURFACES

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Suppose X is a smooth *simply-connected* compact 4-manifold. Let $P = CP^2$ and $Q = -CP^2$ be the complex projective plane with orientation opposite to the usual. We shall say that X is completely decomposable if there exist integers a, b such that X is diffeomorphic to $aP \# bQ$.

By a result of Wall [W1] there always exists an integer k such that $X \# (k+1)P \# kQ$ is completely decomposable. If $X \# P$ is completely decomposable we shall say that X is almost completely decomposable. In [MM1] we demonstrated that any nonsingular hypersurface of CP^3 is almost completely decomposable. In this paper we first announce generalizations of this result in two directions as follows.

THEOREM 1. *Suppose W is a simply-connected nonsingular complex projective 3-fold. Then there exists an integer $m_0 \geq 1$ such that any hypersurface section V_m of W of degree $m \geq m_0$ which is nonsingular will be almost completely decomposable.*

THEOREM 2. *Let V be a nonsingular complex algebraic surface which is a complete intersection. Then V is almost completely decomposable.*

IDEA OF PROOF. The idea of the proofs is to degenerate V (or V_m) to a pair of "less complicated" nonsingular surfaces crossing transversely and then use induction. The topological analysis of such a situation is then taken care of by Corollary 2.5 of [MM2] which states:

COROLLARY. *Suppose W is a compact complex manifold and V, X_1, X_2 are closed complex submanifolds with normal crossing in W . Let $S = X_1 \cap X_2$ and $C = V \cap S$ and suppose as divisors V is linearly equivalent to $X_1 + X_2$. Let $\sigma: X'_2 \rightarrow X_2$ be the monoidal transformation of X_2 with center C . Let S' be the strict image of S in X'_2 and let $T'_2 \rightarrow S', T_1 \rightarrow S$ be tubular neighborhoods of S' in X'_2 and S in X_1 , respectively, with $H_1 = \partial T_1$ and $H'_2 = \partial T'_2$.*

Then there exists a bundle isomorphism $\eta: H'_2 \rightarrow H_2$ which reverses orientation on fibers such that V is diffeomorphic to $X'_2 - T'_2 \cup_{\eta} X_1 - T_1$.

Then if V, X_1, X_2 are simply connected 4-manifolds we can use [M] to

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conclude that $V \approx X_1 \# X_2 \# (n-1)Q \# 2g(P \# Q)$ where $n = \text{card } C$ and $g = \text{genus } S$. The proofs of Theorems 1 and 2 then conclude using an inductive argument.

This method gives us other results for which we establish some additional terminology. A field F is called an algebraic function field of two variables over \mathbf{C} if F is a finitely-generated extension of \mathbf{C} of transcendence degree two. Let \mathcal{F} denote the collection of all such fields. Then for $F \in \mathcal{F}$ there exists a nonsingular algebraic surface whose field of meromorphic functions is F (see [Z]). We shall call any such nonsingular surface a model for F . It is then easy to see that given any two such models V_1, V_2 for F their fundamental groups are isomorphic. Thus we define the fundamental group $\pi_1(F)$ for any $F \in \mathcal{F}$ as the fundamental group of any model V for F . We then let \mathcal{F}_0 be the subcollection of simply-connected F in \mathcal{F} . For $F \in \mathcal{F}_0$ we let $\mu(F) = \inf\{k \mid \text{a model } V \text{ for } F \text{ such that } V \# kP \text{ is completely decomposable}\}$. Using Wall's result previously mentioned, it can be seen that $\mu(F)$ is finite for any $F \in \mathcal{F}_0$. If F is a pure transcendental extension of \mathbf{C} , $\mu(F) = 0$. If $\mu(F) \leq 1$ we shall call F a topologically normal field. We now need

DEFINITION 3. Let $L, K \in \mathcal{F}$. Then L is a satisfactory cyclic extension of K if there exist models V_L, V_K for L , resp. K and a morphism $\Phi: V_L \rightarrow V_K$ with discrete fibers whose ramification locus R_Φ is a nonsingular hypersurface section of V_K whose degree is a multiple of $\deg(\Phi)$.

We then state

THEOREM 4. Let $K \in \mathcal{F}_0$. Then there exists a satisfactory cyclic extension $L \in \mathcal{F}_0$ of K which is of degree 2 over K and topologically normal.

In [M] it is further shown that if K itself is topologically normal then so is any satisfactory cyclic extension. These two results motivate a partial order in \mathcal{F}_0 defined as follows:

For $L, K \in \mathcal{F}_0$ we shall say that L is a satisfactorily resolvable extension of K iff there exists a finite sequence of fields L_0, \dots, L_n in \mathcal{F}_0 with $L_0 = K$, L_{i+1} a satisfactory cyclic extension of L_i and $L_n = L$. We write $K < L$ if L is a satisfactorily resolvable extension of K . Then $<$ induces a partial ordering on \mathcal{F}_0 . Our above results then say that in terms of this partial ordering every sufficiently "large" field L is topologically normal.

Lastly we mention a purely topological counterpart of Theorem 4.

THEOREM 5. Suppose X is a smooth simply-connected 4-manifold. Let $F \in H_2(X, \mathbf{Z})$ with $F^2 \neq 0$ and F divisible by some integer $m \geq 2$. Then there exists a smooth compact simply connected 4-manifold \tilde{X} and a map $\Phi: \tilde{X} \rightarrow X$ exhibiting \tilde{X} as an m -fold branched cover over X whose branch locus R is a nonsingular representative of F such that

- (1) If $F^2 > 0$ then $\tilde{X} \# P$ is completely decomposable.
- (2) If $F^2 < 0$ then $\tilde{X} \# Q$ is completely decomposable.

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