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TRANSFERENCE RESULTS FOR MULTIPLIER OPERATORS

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The purpose of this paper is to show a transference result of the type obtained in [4] and [5] for convolution operators acting on functions defined on Σ_{n-1}, the unit sphere of $\mathbf{R}^n$. As a consequence we obtain a multiplier theorem for expansions in spherical harmonics and Gegenbauer polynomials. Also, Zygmund's inequality for Cesàro sums and that for Littlewood-Paley function g_δ, due to Bonami and Clerc [1], are easily obtained using our results [6]. I wish to express my appreciation to my Ph.D advisors, Professor R. Coifman and G. Weiss, for their encouragement and help in the preparation of this work.

Introduction. Let SO(n) be the group of all rotations of $\mathbf{R}^n$. The left regular representation of SO(n) defined by $R_u f(x) = f(u^{-1}x)$, $u \in \mathrm{SO}(n)$ and $f \in L^2(\Sigma_{n-1})$, decomposes into a direct sum of finite dimensional irreducible representations R^k $(n \geqslant 3)$, $k = 0, 1, \ldots$. $L^2(\Sigma_{n-1}) = \Sigma_{k=0}^\infty H_k$, where H_k, the space of the representation R^k, consists of the spherical harmonics of degree k [2], [8], [9]. If $f \in L^2(\Sigma_{n-1})$, $f(x) = \Sigma_{k=0}^\infty (Z_{e,n-1}^{(k)} * f)(x)$, where $Z_{e,n-1}^{(k)}(x)$ is the zonal spherical harmonic of degree k and pole $e = (0, \ldots, 0, 1)$ and $*$ denotes convolution on Σ_{n-1}. A multiplier M, is an operator that commutes with the action of SO(n) on Σ_{n-1} and is defined on the class P of finite linear combinations of elements in the spaces H_k. Such M assume the form

$$Mf(x) = \sum m_k (Z_{e,n-1}^{(k)} * f)(x) \quad \text{(finite sum)}.$$

Multipliers for expansions in spherical harmonics. Let H be a Hilbert space over the complex numbers and let $L^p(\Sigma_{n-1}, H)$, $1 \leqslant p \leqslant \infty$, be the space of functions $f\colon \Sigma_{n-1} \longrightarrow H$ defined in the usual way replacing absolute values by $\|\cdot\|_H$. For the left regular representation of SO(n) on $L^2(\Sigma_{n-1}, H)$ we have a decomposition entirely similar to the one described above [3]. To a bounded operator on L^2 which commutes with rotations,corresponds a bounded sequence $\{m_k\}_{k=0}^\infty$ of operators on H such that $Mf(x) = \Sigma\, m_k (Z_{e,n-1}^{(k)} * f(x))$ (finite sum) for every $f \in P$. The operator valued function

$$K_r(x) = \sum_{k=0}^\infty r^k Z_{e,n-1}^{(k)}(x) m_k, \qquad r \in [0, 1),$$

is continuous. We write $Mf(x) = \lim_{r \to 1} (K_r * f)(x)$.

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THEOREM 1. *Let M_r be defined on the class P on Σ_{n-2} by letting*

$$M_r g(x) = [(|\sin\theta| K_r(\theta)) * g](x),$$

where $r \in [0, 1)$ and θ is the angle between a variable in Σ_{n-1} and e.

If M_r is bounded uniformly for r close to 1, i.e.

$$\int_{\Sigma_{n-2}} \|M_r g(x)\|_H^p \, dx \leqslant A_p^p \int_{\Sigma_{n-2}} \|g(x)\|_H^p \, dx,$$

$1 \leqslant p < \infty$ and A_p is a constant depending only on p, then

$$\int_{\Sigma_{n-1}} \|Mf(x)\|_H^p \, dx \leqslant A_n^p A_p^p \int_{\Sigma_{n-1}} \|f(x)\|_H^p \, dx.$$

Let $f \in L^1(\mathrm{SO}(n))$; then

$$\int_{\mathrm{SO}(n)} f(u)\, du = c_n \int_{\mathrm{SO}(n-1)} \int_{\mathrm{SO}(n-1)} \int_0^{2\pi} f(\sigma a(\theta)\sigma') |\sin\theta|^{n-2} \, d\theta \, d\sigma \, d\sigma'$$

where du and $d\sigma, d\sigma'$ are the Haar measures of SO(n), SO($n-1$) respectively, and $a(\theta)$ is a rotation by the angle θ in the subspace of $\mathbf{R}^n$ generated by the vectors e and $(0, \ldots, 0, 1, 0)$. Using (2) and the methods of [4] and [5] one obtains the above result.

THEOREM 3. *Let $\{K_j\}_{j=0}^\infty$ be a sequence of integrable zonal functions on Σ_{n-1}. Define the maximal operator $K^*f(x) = \sup_j \|(K_j * f)(x)\|_H$ on $L^p(\Sigma_{n-1}, H)$. If the maximal operator $k^*g(x) = \sup_j \|(|\sin\theta| K_j(\theta)) * g(x)\|_H$ is bounded on $L^p(\Sigma_{n-2}, H)$ with operator norm B_p, then K^* is bounded also and its norm bounded by $A_n B_p$.*

When H is the field of complex numbers we obtain

THEOREM 4. *Let $N = [n/2]$. If the sequence $\{\mathcal{D}^N(m_k)\}_{k=0}^\infty$ defines a bounded multiplier on $L^p(\Sigma_1)$, $1 \leqslant p < \infty$, then $\{m_k\}_{k=0}^\infty$ defines a bounded multiplier on $L^p(\Sigma_n)$, where $\mathcal{D}(m_k) = km_k - (k-2)m_{k-2}$ and $\mathcal{D}^t(m_k) = \mathcal{D}(\mathcal{D}^{t-1}(m_k))$.*

The Marcinkiewicz multiplier theorem [7], together with the above result, give us a multiplier theorem that contains that of Bonami and Clerc [1].

Multipliers for expansions in Gegenbauer polynomials. $L_\lambda^p(-1, 1)$ denotes the space of complex valued measurable functions f on $[-1, 1]$ with respect to the measure $dm_\lambda(x) = (1 - x^2)^{\lambda - 1/2}\, dx$, where $\lambda > 0$ and dx is Lebesgue measure. To each $f \in L_\lambda^p$, we associate the formal sum $f(x) \sim \Sigma_{k=0}^\infty c_k \hat{f}(k) \mathbf{C}_k^\lambda(x)$, where $\mathbf{C}_k^\lambda(x)$ is the normalized Gegenbauer polynomial of order λ, $\mathbf{C}_k^\lambda(1) = 1$, $\hat{f}(k)$ the Fourier coefficient and $c_k^{-1} = \|\mathbf{C}_k^\lambda\|_{2,\lambda}^2$.

A multiplier M assumes the form $Mf(x) \sim \Sigma_{k=0} m_k c_k \hat{f}(k) \mathbf{C}_k^\lambda(x)$, where $\{m\}_{k=0}^\infty$ is a sequence of complex numbers.

THEOREM 5. *Let* λ, δ *be positive real numbers. If the convolution operator with kernel* $g(y)(1-y^2)^\delta$ *is bounded on* L^p_λ *with operator norm* $A_{p,\lambda}$, *then* $g(y)$ *defines a bounded convolution operator on* $L^p_{\lambda+\delta}$ *with norm bounded by* $C_{\beta,\delta}A_{p,\lambda}$.

This theorem implies a transference result for multipliers similar to Theorem 1

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