TRANSFERENCE RESULTS FOR MULTIPLIER OPERATORS

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Communicated by François Treves, January 30, 1976

The purpose of this paper is to show a transference result of the type obtained in [4] and [5] for convolution operators acting on functions defined on $\Sigma_{n-1}$, the unit sphere of $\mathbb{R}^n$. As a consequence we obtain a multiplier theorem for expansions in spherical harmonics and Gegenbauer polynomials. Also, Zygmund’s inequality for Cesáro sums and that for Littlewood-Paley function $g_6$, due to Bonami and Clerc [1], are easily obtained using our results [6]. I wish to express my appreciation to my Ph.D advisors, Professor R. Coifman and G. Weiss, for their encouragement and help in the preparation of this work.

Introduction. Let $SO(n)$ be the group of all rotations of $\mathbb{R}^n$. The left regular representation of $SO(n)$ defined by $R_u f(x) = f(u^{-1}x)$, $u \in SO(n)$ and $f \in L^2(\Sigma_{n-1})$, decomposes into a direct sum of finite dimensional irreducible representations $R^k$ ($n \geq 3$), $k = 0, 1, \ldots$. $L^2(\Sigma_{n-1}) = \sum_{k=0}^{\infty} H_k$, where $H_k$, the space of the representation $R^k$, consists of the spherical harmonics of degree $k$ [2], [8], [9]. If $f \in L^2(\Sigma_{n-1})$, $f(x) = \sum_{k=0}^{\infty} (Z^{(k)}_{e,n-1} * f)(x)$, where $Z^{(k)}_{e,n-1}(x)$ is the zonal spherical harmonic of degree $k$ and pole $e = (0, \ldots, 0, 1)$ and * denotes convolution on $\Sigma_{n-1}$. A multiplier $M$, is an operator that commutes with the action of $SO(n)$ on $\Sigma_{n-1}$ and is defined on the class $P$ of finite linear combinations of elements in the spaces $H_k$. Such $M$ assume the form

$$Mf(x) = \sum m_k (Z^{(k)}_{e,n-1} * f)(x) \quad \text{(finite sum)}.$$

Multipliers for expansions in spherical harmonics. Let $H$ be a Hilbert space over the complex numbers and let $L^p(\Sigma_{n-1}, H)$, $1 \leq p \leq \infty$, be the space of functions $f$: $\Sigma_{n-1} \rightarrow H$ defined in the usual way replacing absolute values by $\| \cdot \|_H$. For the left regular representation of $SO(n)$ on $L^2(\Sigma_{n-1}, H)$ we have a decomposition entirely similar to the one described above [3]. To a bounded operator on $L^2$ which commutes with rotations, corresponds a bounded sequence $\{m_k\}_{k=0}^{\infty}$ of operators on $H$ such that $Mf(x) = \sum m_k (Z^{(k)}_{e,n-1} * f)(x)$ (finite sum) for every $f \in P$. The operator valued function

$$K_r(x) = \sum_{k=0}^{\infty} r^k Z^{(k)}_{e,n-1} (x)m_k, \quad r \in [0, 1],$$

is continuous. We write $Mf(x) = \lim_{r \rightarrow 1} (K_r * f)(x)$.


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THEOREM 1. Let $M_r$ be defined on the class $P$ on $\Sigma_{n-2}$ by letting

$$M_r g(x) = [(|\sin \theta| K_r(\theta)) \ast g](x),$$

where $r \in [0, 1)$ and $\theta$ is the angle between a variable in $\Sigma_{n-1}$ and $e$.

If $M_r$ is bounded uniformly for $r$ close to 1, i.e.

$$\int_{\Sigma_{n-2}} \|M_r g(x)\|_p^p dx \leq A_p \int_{\Sigma_{n-2}} \|g(x)\|_p^p dx,$$

$1 < p < \infty$ and $A_p$ is a constant depending only on $p$, then

$$\int_{\Sigma_{n-1}} \|Mf(x)\|_p^p dx \leq A_p A_p \int_{\Sigma_{n-1}} \|f(x)\|_p^p dx.$$

Let $f \in L^1(SO(n))$; then

$$\int_{SO(n)} f(u) du = c_n \int_{SO(n-1)} \int_{SO(n-1)} \int_0^{2\pi} f(\sigma a(\theta) a') |\sin \theta|^{n-2} d\theta \, d\sigma \, d\sigma'$$

where $du$ and $d\sigma, d\sigma'$ are the Haar measures of $SO(n), SO(n-1)$ respectively, and $a(\theta)$ is a rotation by the angle $\theta$ in the subspace of $\mathbb{R}^n$ generated by the vectors $e$ and $(0, \ldots, 0, 1, 0)$. Using (2) and the methods of [4] and [5] one obtains the above result.

THEOREM 3. Let $\{K_j\}_{j=0}^\infty$ be a sequence of integrable zonal functions on $\Sigma_{n-1}$. Define the maximal operator $K^* f(x) = \sup_j \|K_j f\|_H$ on $L^p(\Sigma_{n-1}, H)$. If the maximal operator $K^* g(x) = \sup_j \|(|\sin \theta| K_j(\theta)) \ast g(x)\|_H$ is bounded on $L^p(\Sigma_{n-2}, H)$ with operator norm $B$, then $K^*$ is bounded also and its norm bounded by $A p B p$.

When $H$ is the field of complex numbers we obtain

THEOREM 4. Let $N = [n/2]$. If the sequence $\{\mathcal{D}^N (m_k)\}_{k=0}^\infty$ defines a bounded multiplier on $L^p(\Sigma_1)$, $1 < p < \infty$, then $\{m_k\}_{k=0}^\infty$ defines a bounded multiplier on $L^p(\Sigma_n)$, where $\mathcal{D} (m_k) = km_k - (k - 2)m_{k-2}$ and $\mathcal{D}^1 (m_k) = 0$. The Marcinkiewicz multiplier theorem [7], together with the above result, give us a multiplier theorem that contains that of Bonami and Clerc [1].

Multipliers for expansions in Gegenbauer polynomials. $L^p_\lambda(-1, 1)$ denotes the space of complex valued measurable functions $f$ on $[-1, 1]$ with respect to the measure $dm_\lambda(x) = (1 - x^2)^{\lambda - 3/2} dx$, where $\lambda > 0$ and $dx$ is Lebesgue measure. To each $f \in L^p_\lambda$, we associate the formal sum $f(x) \sim \sum_{k=0}^\infty c_k \hat{f}(k) C^\lambda_k(x)$, where $C^\lambda_k(x)$ is the normalized Gegenbauer polynomial of order $\lambda$, $C^\lambda_k(1) = 1$, $\hat{f}(k)$ the Fourier coefficient and $c_k^{-1} = \|C_k^\lambda\|^2_2$. A multiplier $M$ assumes the form $Mf(x) \sim \sum_{k=0}^\infty m_k c_k \hat{f}(k) C^\lambda_k(x)$, where $\{m_k\}_{k=0}^\infty$ is a sequence of complex numbers.
Theorem 5. Let $\lambda, \delta$ be positive real numbers. If the convolution operator with kernel $g(y)(1 - y^2)^{\delta}$ is bounded on $L^p_\lambda$ with operator norm $A_{p,\lambda}$, then $g(y)$ defines a bounded convolution operator on $L^p_{\lambda+\delta}$ with norm bounded by $C_{\beta,\delta}A_{p,\lambda}$.

This theorem implies a transference result for multipliers similar to Theorem 1.

REFERENCES


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