BOUNDS ON THE EIGENVALUES OF THE LAPLACE AND SCHROEDINGER OPERATORS

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If $\Omega$ is an open set in $\mathbb{R}^n$, and if $\tilde{N}(\Omega, \lambda)$ is the number of eigenvalues of $-\Delta$ (with Dirichlet boundary conditions on $\partial \Omega$) which are $\leq \lambda$ ($\lambda \geq 0$), one has the asymptotic formula of Weyl [1], [2]:

$$\lim_{\lambda \to \infty} \lambda^{-n/2} \tilde{N}(\Omega, \lambda) = C_n |\Omega|.$$  

Here $|\Omega|$ is the volume of $\Omega$ and $C_n = (4\pi)^{-n/2} \Gamma(1 + n/2)^{-1}$. The same holds [3] if $\mathbb{R}^n$ is replaced by a Riemannian manifold, $M$, with $|\Omega|$ being the Riemannian volume and $\Delta$ being the Laplace-Beltrami operator. One purpose of this note is to state that there often exist bounds of the form

$$(1a) \quad \tilde{N}(\Omega, \lambda) \leq D_n \lambda^{n/2} |\Omega|, \quad \forall \lambda \geq 0, \forall \Omega \subset M,$$

$$(1b) \quad \tilde{N}(\Omega, \lambda) \leq (D_n \lambda^{n/2} + E_n) |\Omega|, \quad \forall \lambda \geq 0, \forall \Omega \subset M,$$

with $D_n, E_n$ independent of $\lambda$ and $\Omega$ and depending only on $M$. (1a) holds for noncompact $M$ if condition (8), below, holds. In particular, (1a) holds for $\mathbb{R}^n$ and for homogeneous spaces with curvature $\leq 0$. (1b) always holds for compact $M$, and it also holds for noncompact $M$ if condition (9) holds.

**Remark.** There is an asymptotic formula [4], [5]:

$$\tilde{N}(\Omega, \lambda) = C_n \lambda^{n/2} |\Omega| + O(\lambda^{(n-1)/2}).$$

While this has the correct limiting constant, $C_n$, the remainder, $O(\cdot)$, can get very large if $\Omega$ is very irregular. The remainder is not bounded by a universal constant times $|\Omega| \lambda^{(n-1)/2}$ or even $|\Omega| \lambda^{n/2}$. Our emphasis is different. By introducing $D_n \geq C_n$ we have a bound which is universal in the sense that it depends on $M$ but not on $\Omega \subset M$; in particular, (1) is applicable to unbounded $\Omega$.

A second, closely related problem is to estimate $N_{\alpha}(V) = \text{number of non-negative eigenvalues of } -\Delta + V(x)$ on $L^2(M)$ which are $\leq \alpha \leq 0$. There exists an asymptotic formula [6], [7], [8] for suitably regular $V$:

$$\lim_{\gamma \to \infty} \gamma^{-n/2} N_{\gamma \alpha}(\gamma V) = C_n \int_M [V(x) - \alpha]^{n/2} dx$$

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where $\lambda = \frac{1}{2}(|V|-V)$, and $dx$ is the Riemannian volume element. Our new, nonasymptotic result is

$$N_{\alpha}(V) \leq L_n \int_M |V(x) - \alpha|^{n/2} dx, \ \forall \alpha, V$$

when $M$ satisfies (8) and $\dim(M) \geq 3$. [(3) was obtained simultaneously and independently by M. Cwikel [9]; his estimate for $L_n$ is not as sharp as ours, however. When $n = 3$, our $L_3 = 0.116$ and it is known that $L_3 > 0.078$.]

The connection between $\tilde{N}(\Omega, \lambda)$ and $N_{\alpha}(V)$ is the following elementary consequence of the min-max principle:

$$\tilde{N}(\Omega, \lambda) \leq N_{\alpha}((\alpha - \lambda)\chi_{\Omega}), \ \forall \alpha \leq 0$$

where $\chi_{\Omega}$ is the characteristic function of $\Omega$. Thus (3) implies $\tilde{N}(\Omega, \lambda) \leq L_n\lambda^{n/2} |\Omega|$ for $\dim(M) \geq 3$. Another important consequence of the min-max principle is

$$N_{\alpha}(V) \leq N_{\alpha+\beta}(-(V + \beta)_{-}), \ 0 \leq \beta \leq -\alpha.$$ 

Consequently, one need consider only the case $V = -V_-$ in (3).

The asymptotic formula (2) has been extended to $V_- \in L^{n/2+\epsilon} \cap L^{n/2-\epsilon}$ by Simon [10]. Using his methods and (3), one easily extends (2) to all $V_- \in L^{n/2}$.

Results (1) and (3) are corollaries of the following

**Theorem.** Let $f: [0, \infty) \rightarrow [0, \infty)$ be convex and polynomially bounded at infinity and satisfy

$$\int_0^\infty t^{-1}e^{-tf(t)}dt = 1.$$ 

For $t > 0$, let $G(x, y; t)$ be the kernel of $e^{t\Delta}$, i.e. the fundamental solution of the heat equation on the Riemannian manifold $M$. Then, for $\alpha \leq 0$,

$$N_{\alpha}(V) \leq \int_M dx \int_0^\infty t^{-1}e^{at}G(x, x; t)f(tV_-(x))dt.$$ 

Our proof of this theorem uses the Wiener integral in an essential way and will be published elsewhere.

To apply (7) we choose $f(t) = 0, t \leq a, f(t) = b(t-a), t \geq a$, for some $a, b > 0$ such that (6) holds. To prove (3), we assume

$$G(x, x; t) \leq At^{-n/2}, \ \forall x \in M, \forall t > 0.$$ 

This holds for $R^n$ ($A = (4\pi)^{-n/2}$) and for homogeneous spaces with curvature $\leq 0$. Next we use (5) with $\beta = -\alpha$ and then (7) with $\alpha = 0$.

To prove (1a) we assume (8). For (1b) we require a bound of the form

$$G(x, x; t) \leq At^{-n/2} + B, \ \forall x \in M, \forall t > 0,$$

which always holds for compact $M$, for example. In either case, using (4) and (7) with $\alpha = \lambda$. 

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\[ \tilde{\mathcal{N}}(\Omega, \lambda) \leq \int_{\Omega} dx \int_0^\infty t^{-1/2} e^{-t/2} G(x, x; (2\lambda)^{-1} t) f(t) dt. \]

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**ADDED IN PROOF.** I have recently become aware of the paper of G. V. Rozenbljum, Dokl. Akad. Sci. SSSR 202 no. 5 (1972), 1012–1015 (MR 45 #4216) in which a proof of (3) is announced. Rozenbljum’s method is completely different, however, and his estimate for \( L_n \) does not appear to be as sharp.

**REFERENCES**


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