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BOUNDS ON THE EIGENVALUES OF THE LAPLACE AND SCHROEDINGER OPERATORS

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If Ω is an open set in $\mathbf{R}^n$, and if $\widetilde{N}(\Omega, \lambda)$ is the number of eigenvalues of $-\Delta$ (with Dirichlet boundary conditions on $\partial\Omega$) which are $\leqslant \lambda$ ($\lambda \geqslant 0$), one has the *asymptotic* formula of Weyl [1], [2]: $\lim_{\lambda\to\infty} \lambda^{-n/2}\widetilde{N}(\Omega, \lambda) = C_n|\Omega|$. Here $|\Omega|$ is the volume of Ω and $C_n = (4\pi)^{-n/2}\Gamma(1 + n/2)^{-1}$. The same holds [3] if $\mathbf{R}^n$ is replaced by a Riemannian manifold, M, with $|\Omega|$ being the Riemannian volume and Δ being the Laplace-Beltrami operator. One purpose of this note is to state that there often exist bounds of the form

$$\widetilde{N}(\Omega, \lambda) \leqslant D_n\lambda^{n/2}|\Omega|, \forall\lambda \geqslant 0, \forall\Omega \subset M, \tag{1a}$$

$$\widetilde{N}(\Omega, \lambda) \leqslant (D_n\lambda^{n/2} + E_n)|\Omega|, \quad \forall\lambda \geqslant 0, \forall\Omega \subset M, \tag{1b}$$

with D_n, E_n *independent of* λ *and* Ω and depending only on M. (1a) holds for noncompact M if condition (8), below, holds. In particular, (1a) holds for $\mathbf{R}^n$ and for homogeneous spaces with curvature $\leqslant 0$. (1b) always holds for compact M, and it also holds for noncompact M if condition (9) holds.

REMARK. There is an asymptotic formula [4], [5]: $\widetilde{N}(\Omega, \lambda) = C_n\lambda^{n/2}|\Omega| + O(\lambda^{(n-1)/2})$. While this has the correct limiting constant, C_n, the remainder, $O(\cdot)$, can get very large if Ω is very irregular. The remainder is not bounded by a universal constant times $|\Omega|\lambda^{(n-1)/2}$ or even $|\Omega|\lambda^{n/2}$. Our emphasis is different. By introducing $D_n \geqslant C_n$ we have a bound which is universal in the sense that it depends on M but *not* on $\Omega \subset M$; in particular, (1) is applicable to unbounded Ω.

A second, closely related problem is to estimate $N_\alpha(V)$ = number of nonpositive eigenvalues of the Schroedinger operator $-\Delta + V(x)$ on $L^2(M)$ which are $\leqslant \alpha \leqslant 0$. There exists an asymptotic formula [6], [7], [8] for suitably regular V:

$$\lim_{\gamma\to\infty} \gamma^{-n/2}N_{\gamma\alpha}(\gamma V) = C_n \int_M [V(x) - \alpha]_-^{n/2}\, dx \tag{2}$$

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where $V_- = \frac{1}{2}(|V| - V)$, and dx is the Riemannian volume element. Our new, *nonasymptotic result* is

$$N_\alpha(V) \leqslant L_n \int_M [V(x) - \alpha]_-^{n/2}\, dx, \quad \forall \alpha, V \tag{3}$$

when M satisfies (8) and $\dim(M) \geqslant 3$. [(3) was obtained simultaneously and independently by M. Cwikel [**9**]; his estimate for L_n is not as sharp as ours, however. When $n = 3$, our $L_3 = .116$ and it is known that $L_3 \geqslant .078$.]

The connection between $\widetilde{N}(\Omega, \lambda)$ and $N_\alpha(V)$ is the following elementary consequence of the min-max principle:

$$\widetilde{N}(\Omega, \lambda) \leqslant N_\alpha((\alpha - \lambda)\chi_\Omega), \quad \forall \alpha \leqslant 0 \tag{4}$$

where χ_Ω is the characteristic function of Ω. Thus (3) implies $\widetilde{N}(\Omega, \lambda) \leqslant L_n \lambda^{n/2} |\Omega|$ for $\dim(M) \geqslant 3$. Another important consequence of the min-max principle is

$$N_\alpha(V) \leqslant N_{\alpha+\beta}(-(V + \beta)_-), \qquad 0 \leqslant \beta \leqslant -\alpha. \tag{5}$$

Consequently, one need consider only the case $V = -V_-$ in (3).

The asymptotic formula (2) has been extended to $V_- \in L^{n/2+\epsilon} \cap L^{n/2-\epsilon}$ by Simon [**10**]. Using his methods and (3), one easily extends (2) to all $V_- \in L^{n/2}$.

Results (1) and (3) are corollaries of the following

THEOREM. *Let f: $[0, \infty) \longrightarrow [0, \infty)$ be convex and polynomially bounded at infinity and satisfy*

$$\int_0^\infty t^{-1} e^{-t} f(t)\, dt = 1. \tag{6}$$

For $t > 0$, let $G(x, y; t)$ be the kernel of $e^{t\Delta}$, i.e. the fundamental solution of the heat equation on the Riemannian manifold M. Then, for $\alpha \leqslant 0$,

$$N_\alpha(V) \leqslant \int_M dx \int_0^\infty t^{-1} e^{\alpha t} G(x, x; t) f(tV_-(x))\, dt. \tag{7}$$

Our proof of this theorem uses the Wiener integral in an essential way and will be published elsewhere.

To apply (7) we choose $f(t) = 0$, $t \leqslant a$, $f(t) = b(t - a)$, $t \geqslant a$, for some $a, b > 0$ such that (6) holds. To prove (3), we assume

$$G(x, x; t) \leqslant At^{-n/2}, \quad \forall x \in M, \forall t > 0. \tag{8}$$

This holds for $\mathbf{R}^n$ $(A = (4\pi)^{-n/2})$ and for homogeneous spaces with curvature $\leqslant 0$. Next we use (5) with $\beta = -\alpha$ and then (7) with $\alpha = 0$.

To prove (1a) we assume (8). For (1b) we require a bound of the form

$$G(x, x; t) \leqslant At^{-n/2} + B, \quad \forall x \in M, \forall t > 0, \tag{9}$$

which always holds for compact M, for example. In either case, using (4) and (7) with $\alpha = -\lambda$,

$$\widetilde{N}(\Omega, \lambda) \leqslant \int_\Omega dx \int_0^\infty t^{-1} e^{-t/2} G(x, x; (2\lambda)^{-1} t) f(t)\, dt.$$

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ADDED IN PROOF. I have recently become aware of the paper of G. V. Rozenbljum, Dokl. Akad. Sci. SSSR 202 no. 5 (1972), 1012–1015 (MR 45 #4216) in which a proof of (3) is announced. Rozenbljum's method is completely different, however, and his estimate for L_n does not appear to be as sharp.

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