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SEIFERT FIBERED SPACES IN IRREDUCIBLE, SUFFICIENTLY-LARGE 3-MANIFOLDS

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In [2], F. Waldhausen announced theorems about singular annuli and tori in a bounded, orientable, irreducible 3-manifold M, analogous to the Dehn Lemma-Loop Theorem for singular disks and the Sphere Theorem for singular spheres. The Torus Theorem was used to prove that the centralizer of an element of $\pi_1(M)$ is finitely generated; as a corollary, this yields a simple proof of the earlier result that $\pi_1(M)$ has no infinitely divisible elements.

As consequences of the main theorem announced below, we obtain greatly sharpened versions of all the results mentioned above. We also obtain a canonical way of decomposing a rather general compact 3-manifold into submanifolds with nice properties.

Our main theorem is a homotopy-classification theorem for certain maps of Seifert fibered spaces and I-bundles into the 3-manifold M. An immediate consequence of the theorem, in effect a special case, is a homotopy-classification of singular annuli and tori in M. Similar results have been obtained independently by Johannson [1].

In what follows, manifolds are understood to be piecewise-linear. A Seifert fibered space or an I-bundle over a surface is understood to have a fixed fibration.

DEFINITIONS. A *3-manifold pair* is a pair (M, T), where M is a 3-manifold and $T \subset \partial M$ is a 2-manifold. The pair (M, T) is *sufficiently-large* if M is compact, connected, orientable, irreducible and sufficiently large while T is compact and each component of T is incompressible. A *Seifert pair* is a 3-manifold pair $(\mathcal{S}, \mathcal{F})$, in which both $\mathcal{S}$ and $\mathcal{F}$ are compact and orientable, and such that for each component S of $\mathcal{S}$, there exists either (i) a homeomorphism of S onto a Seifert fibered space, which maps $S \cap \mathcal{F}$ onto a union of fibers, or (ii) a homeomorphism of S onto a PL I-bundle over a surface, which maps $S \cap \mathcal{F}$ onto the associated ∂I-bundle. The Seifert pair (S, F), with S connected, is called *degenerate* if either (i) $\pi_1(S) = \{1\}$, or (ii) $F = \varnothing$ and $\pi_1(S)$ is cyclic.

Let (S, F) be a polyhedral pair such that S is connected. A map of pairs

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$f: (S, F) \longrightarrow (M, T)$, where (M, T) is a 3-manifold pair, is *essential* if (i) $f_*: \pi_1(S) \longrightarrow \pi_1(M)$ is a monomorphism, and (ii) f cannot be homotoped as a map of pairs to a map f' such that $f'(S) \subset T$.

MAIN THEOREM. *Let (M, T) be a sufficiently-large 3-manifold pair. Then there exists a Seifert pair (Σ, Φ), where $\Sigma \subset M$, $\Phi \subset T$, such that for any non-degenerate, connected Seifert pair (S, F) and any essential map $f: (S, F) \longrightarrow (M, T)$, f is homotopic as a map of pairs to a map g such that $g(S) \subset \Sigma$ and $g(F) \subset \Phi$.*

It is immediate from this theorem that any essential map of $(S^1 \times I, S^1 \times \partial I)$ or $(S^1 \times S^1, \varnothing)$ into (M, T) is homotopic (as a map of pairs) to a map into (Σ, Φ). This version of the Torus-annulus Theorem may be used to prove Waldhausen's Theorem with no great difficulty.

THEOREM. *Let M be a compact, connected, orientable, irreducible and sufficiently large 3-manifold. Then there exists a collection (possibly empty) of disjoint, compact Seifert fibered spaces $\Sigma_1, \ldots, \Sigma_k \subset M$ with incompressible boundaries, and subgroups (well defined up to conjugacy) G_i of index $\leqslant 2$ in $\overline{G}_i$ $\mathrm{Im}(\pi_1(\Sigma_i) \longrightarrow \pi_1(M))$, such that*: (i) *each G_i is the centralizer of some element of $\pi_1(M)$, but no G_i is cyclic*; (ii) *any noncyclic subgroup of $\pi_1(M)$ which is the centralizer of an element of $\pi_1(M)$ is conjugate in $\pi_1(M)$ to a subgroup of one of the groups $\overline{G}_i$*; (iii) *any subgroup of $\pi_1(M)$ which is the centralizer of an element of $\pi_1(M)$, and has no abelian subgroup of index $\leqslant 2$, is conjugate in $\pi_1(M)$ to one of the G_i.*

The set of all roots of an element of the fundamental group of a Seifert fibered 3-manifold is easily described; and by combining this description with the above theorem on centralizers, one may obtain a description of the set of roots of any $x \in \pi_1(M)$ when M is sufficiently large. In the case of a knot group, this answers a question of L. P. Neuwirth, and it is in this case that we shall state the result.

THEOREM. *Let K be a polyhedral knot in S^3 and let M denote its associated knot space. Then there exists a collection (possibly empty) of disjoint torus knot spaces*[2] *$T_1, \ldots, T_k \subset M$ with incompressible boundaries, and having the following properties.* (i) *Any element of $\pi_1(M)$, the roots of which do not lie in a single subgroup of $\pi_1(M)$, is conjugate to an element of one of the subgroups (well defined up to conjugacy) $G_i = \mathrm{Im}(\pi_1(T_i) \longrightarrow \pi_1(M))$*; (ii) *the roots of any element of G_i $(1 \leqslant i \leqslant k)$ all lie in G_i.*

We call a compact, orientable, irreducible, sufficiently large 3-manifold M atoroidal if M contains no essentially embedded annuli or tori.

[2] ADDED IN PROOF. Each T_i is homeomorphic to the residual space of a standard torus knot *either* in S^3 *or* in standard solid torus in S^3.

DECOMPOSITION THEOREM. *Let* (M, T) *be a sufficiently-large* 3-*manifold pair. Then* $M = N \cup \Sigma$ *where* (i) Σ *is the first term of a Seifert pair* $(\Sigma, \Phi) \subset (M, T)$ *where the inclusion map of* (Σ, Φ) *into* (M, T) *is essential, and* N *is atoroidal*; (ii) $N \cap \Sigma = \partial N \cap \partial \Sigma$ *is a union of incompressible annuli and tori*; (iii) *if* (S, F) *is a nondegenerate Seifert pair and* $f\colon (S, F) \longrightarrow (M, T)$ *is an essential map, then* f *can be deformed as a map of pairs to a map* $g\colon (S, F) \longrightarrow (M, T)$ *with* $g(S) \subset \Sigma$, $g(F) \subset T$; *and* (iv) *no proper subcollection* (Σ', Φ') *of components of* (Σ, Φ) *satisfies* (i)–(iii).

It is not difficult to show that the pair (Σ, Φ) satisfying the above is unique. We have conjectured that if N is an atoroidal, sufficiently large 3-manifold, then $\pi_1(N)$ completely determines N up to homeomorphism. A positive solution to this has been announced in [1].

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