In [2], F. Waldhausen announced theorems about singular annuli and tori in a bounded, orientable, irreducible 3-manifold $M$, analogous to the Dehn Lemma-Loop Theorem for singular disks and the Sphere Theorem for singular spheres. The Torus Theorem was used to prove that the centralizer of an element of $\pi_1(M)$ is finitely generated; as a corollary, this yields a simple proof of the earlier result that $\pi_1(M)$ has no infinitely divisible elements.

As consequences of the main theorem announced below, we obtain greatly sharpened versions of all the results mentioned above. We also obtain a canonical way of decomposing a rather general compact 3-manifold into submanifolds with nice properties.

Our main theorem is a homotopy-classification theorem for certain maps of Seifert fibered spaces and $I$-bundles into the 3-manifold $M$. An immediate consequence of the theorem, in effect a special case, is a homotopy-classification of singular annuli and tori in $M$. Similar results have been obtained independently by Johannson [1].

In what follows, manifolds are understood to be piecewise-linear. A Seifert fibered space or an $I$-bundle over a surface is understood to have a fixed fibration.

**Definitions.** A 3-manifold pair is a pair $(M, T)$, where $M$ is a 3-manifold and $T \subset \partial M$ is a 2-manifold. The pair $(M, T)$ is sufficiently-large if $M$ is compact, connected, orientable, irreducible and sufficiently large while $T$ is compact and each component of $T$ is incompressible. A Seifert pair is a 3-manifold pair $(S, F)$, in which both $S$ and $F$ are compact and orientable, and such that for each component $S$ of $S$, there exists either (i) a homeomorphism of $S$ onto a Seifert fibered space, which maps $S \cap F$ onto a union of fibers, or (ii) a homeomorphism of $S$ onto a PL $I$-bundle over a surface, which maps $S \cap F$ onto the associated $\partial I$-bundle. The Seifert pair $(S, F)$, with $S$ connected, is called degenerate if either (i) $\pi_1(S) = \{1\}$, or (ii) $F = \emptyset$ and $\pi_1(S)$ is cyclic.

Let $(S, F)$ be a polyhedral pair such that $S$ is connected. A map of pairs

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f: (S, F) → (M, T), where (M, T) is a 3-manifold pair, is essential if (i) \( f_\#: \pi_1(S) \to \pi_1(M) \) is a monomorphism, and (ii) f cannot be homotoped as a map of pairs to a map \( f' \) such that \( f'(S) \subset T \).

**Main Theorem.** Let \((M, T)\) be a sufficiently-large 3-manifold pair. Then there exists a Seifert pair \((\Sigma, \Phi)\), where \( \Sigma \subset M, \Phi \subset T \), such that for any non-degenerate, connected Seifert pair \((S, F)\) and any essential map \( f: (S, F) \to (M, T) \), \( f \) is homotopic as a map of pairs to a map \( g \) such that \( g(S) \subset \Sigma \) and \( g(F) \subset \Phi \).

It is immediate from this theorem that any essential map of \((S^1 \times I, S^1 \times \partial I)\) or \((S^1 \times S^1, \emptyset)\) into \((M, T)\) is homotopic (as a map of pairs) to a map into \((\Sigma, \Phi)\). This version of the Torus-annulus Theorem may be used to prove Waldhausen's Theorem with no great difficulty.

**Theorem.** Let \( M \) be a compact, connected, orientable, irreducible and sufficiently large 3-manifold. Then there exists a collection (possibly empty) of disjoint, compact Seifert fibered spaces \( \Sigma_1, \ldots, \Sigma_k \subset M \) with incompressible boundaries, and subgroups (well defined up to conjugacy) \( G_i \) of index \(< 2 \) in \( \overline{\pi_1(M)} \), such that: (i) each \( G_i \) is the centralizer of some element of \( \pi_1(M) \), but no \( G_i \) is cyclic; (ii) any noncyclic subgroup of \( \pi_1(M) \) which is the centralizer of an element of \( \pi_1(M) \) is conjugate in \( \pi_1(M) \) to a subgroup of one of the groups \( G_i \); (iii) any subgroup of \( \pi_1(M) \) which is the centralizer of an element of \( \pi_1(M) \), and has no abelian subgroup of index \(< 2 \), is conjugate in \( \pi_1(M) \) to one of the \( G_i \).

The set of all roots of an element of the fundamental group of a Seifert fibered 3-manifold is easily described; and by combining this description with the above theorem on centralizers, one may obtain a description of the set of roots of any \( x \in \pi_1(M) \) when \( M \) is sufficiently large. In the case of a knot group, this answers a question of L. P. Neuwirth, and it is in this case that we shall state the result.

**Theorem.** Let \( K \) be a polyhedral knot in \( S^3 \) and let \( M \) denote its associated knot space. Then there exists a collection (possibly empty) of disjoint torus knot spaces \( T_1, \ldots, T_k \subset M \) with incompressible boundaries, and having the following properties, (i) Any element of \( \pi_1(M) \), the roots of which do not lie in a single subgroup of \( \pi_1(M) \), is conjugate to an element of one of the subgroups (well defined up to conjugacy) \( G_i = \text{Im}(\pi_1(T_i) \to \pi_1(M)) \); (ii) the roots of any element of \( G_i \) \((1 \leq i \leq k)\) all lie in \( G_i \).

We call a compact, orientable, irreducible, sufficiently large 3-manifold \( M \) atoroidal if \( M \) contains no essentially embedded annuli or tori.

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\(^2\)Added in proof. Each \( T_i \) is homeomorphic to the residual space of a standard torus knot either in \( S^3 \) or in standard solid torus in \( S^3 \).
DECOMPOSITION THEOREM. Let \((M, T)\) be a sufficiently-large 3-manifold pair. Then \(M = N \cup \Sigma\) where (i) \(\Sigma\) is the first term of a Seifert pair \((\Sigma, \Phi) \subset (M, T)\) where the inclusion map of \((\Sigma, \Phi)\) into \((M, T)\) is essential, and \(N\) is atoroidal; (ii) \(N \cap \Sigma = \partial N \cap \partial \Sigma\) is a union of incompressible annuli and tori; (iii) if \((S, F)\) is a nondegenerate Seifert pair and \(f: (S, F) \to (M, T)\) is an essential map, then \(f\) can be deformed as a map of pairs to a map \(g: (S, F) \to (M, T)\) with \(g(S) \subset \Sigma, g(F) \subset T\); and (iv) no proper subcollection \((\Sigma', \Phi')\) of components of \((\Sigma, \Phi)\) satisfies (i)–(iii).

It is not difficult to show that the pair \((\Sigma, \Phi)\) satisfying the above is unique. We have conjectured that if \(N\) is an atoroidal, sufficiently large 3-manifold, then \(\pi_1(N)\) completely determines \(N\) up to homeomorphism. A positive solution to this has been announced in [1].

REFERENCES


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