Throughout, $R = (V(R), E(R))$ is a finite loopless digraph with vertex set $V(R)$ and edge set $E(R) \subseteq V(R) \times V(R)$, which may contain cycles. Let $F(u) = \{v \in V(R): (u, v) \in E(R)\}$, $Z = \text{nonnegative integers}$, $GF(2)^n$ the $n$-fold cartesian product of $GF(2)$.

Put any number of stones on distinct vertices of $R$. Two players play alternately. Each player at his turn moves one stone from a vertex $u$ to some $v \in F(u)$. If $v$ was occupied, both stones get removed (annihilation). The player making the last move wins. If there is no last move, the game is a tie.

Such an annihilation game belongs to a large class of combinatorial games discussed in [1], [3], which are analyzable by the Generalized Sprague-Grundy Function (GSG-function) $G: V(R) \rightarrow Z \cup \{\infty\}$ [1], [2], [3] with associated counter function $c: V f(R) \rightarrow Z$ where $V f(R) = \{u \in V(R): G(u) < \infty\}$ [2]. Here $R$ is the game-graph of the game.

Our main result is the construction of a complete strategy for the game, which is polynomial in $n = |V(R)|$.

Let $C(R)$ be the game-graph of the annihilation game on $R$, also called the contrajunctive compound of $R$. If $V(R) = \{u_1, \ldots, u_n\}$, the vertices of $V(C(R))$ (game positions) constitute the set of all $n$-tuples $\bar{u} = (\alpha_1, \ldots, \alpha_n)$ over $GF(2)$, where $\alpha_i = 1$ if and only if a stone is on $u_i$. Also $(\bar{u}, \bar{v}) \in E(C(R))$ if and only if there is a move from $\bar{u}$ to $\bar{v}$. Thus $V(C(R))$ is identical with the linear space $GF(2)^n$ under the operation $\oplus, \Sigma'$ of Nim-sum (below: Generalized Nim-sum [1], [3]).

**Lemma 1.** Let

$$C^f(R) = \{\bar{u} \in V(C(R)): G(\bar{u}) < \infty\}, \quad C_i(R) = \{\bar{u} \in V(C(R)): G(\bar{u}) = i < \infty\}. $$

Then

(i) $C^f(R)$ is a linear subspace of $V(C(R))$.
(ii) $G$ is a homomorphism from $C^f(R)$ onto $GF(2)^t$ with kernel $C_0(R)$ ($t = O(\log_2 n)$). In fact,

$$G(\bar{u}) < \infty \Rightarrow G(\bar{u} \oplus \bar{v}) = G(\bar{u}) \oplus G(\bar{v}).$$

(iii) $\{C_i(R): 0 \leq i < 2^t\} = C^f(R)/C_0(R)$. 

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Let $L^k_i(R) = \{ \tilde{u} \in C_i(R) : \|\tilde{u}\| = k \}$, $LF^k_i(R) = \{ \tilde{u} \in C_i(R) : \|\tilde{u}\| = k \}$, 
\[ \mathcal{V}(S) = \text{linear span of } S, \mathcal{S}_0(R) = L^4_0(R) \cup L^2_0(R) \cup L^1_0(R), \mathcal{S}^f(R) = \mathcal{S}_0(R) \cup LF^2(R). \]

**Lemma 2.** 
(i) $C_0(R) = \mathcal{V}(\mathcal{S}_0(R))$.
(ii) $C^f(R) = \mathcal{V}(\mathcal{S}^f(R))$.
(iii) There exists a basis $\beta^f = (\tilde{u}_1, \ldots, \tilde{u}_m, \tilde{v}_1, \ldots, \tilde{v}_t)$ for $C^f(R)$ such that $\beta_0 = (\tilde{u}_1, \ldots, \tilde{u}_m)$ is a basis of $C_0(R)$ and $\tilde{v}_i \in L^2_{j(i)}(R)$, where $j(i) = 2^{t-1}$ $(1 \leq i \leq t)$.

**Note.** For $m \geq 0$, denote by $C^{(m)}(R)$ the subgraph of $C(R)$ with vertices $\tilde{u}$ satisfying $\|\tilde{u}\| < m$. Then $C^{(m)}(R)$ has $O(n^m)$ vertices, and $\tilde{u} \in V(C^{(m)}(R)) \Rightarrow F(\tilde{u}) \subset V(C^{(m)}(R))$. Hence $G$ on $C^{(m)}(R)$ can be computed from $C^{(m)}(R)$ alone. In particular, $\mathcal{S}^f(R) \subset V(C^{(4)}(R))$. Hence $\mathcal{S}^f(R)$ can be computed in $O(n^6)$ steps using standard algorithms for computing the GSG-function [1].

**Theorem 1.** There exists an $n \times n$ matrix $\Gamma$ over $GF(2)$ which can be computed polynomially, such that for every $\tilde{u} \in V(C(R))$ we have $\Gamma \cdot \tilde{u}' = (e_1, \ldots, e_n)'$, where

\[ \tilde{u} = \sum_{i=1}^m e_i \tilde{u}_i + \sum_{j=1}^t e_{m+j} \tilde{v}_j \oplus \sum_{k=1}^{n-m-t} e_{m+t+k} \tilde{z}_k, \]

and $(\tilde{z}_1, \ldots, \tilde{z}_k)$ is a basis of a complementary space of $C^f(R)$. Moreover, letting $Q(\tilde{u}) = (e_n, e_{n-1}, \ldots, e_{m+1})$, $Q$ is a homomorphism from $V(C(R))$ onto $GF(2)^{n-m}$ with kernel $C_0(R)$, such that $G(\tilde{u}) = \tilde{Q}(\tilde{u}) = \Sigma_{i=1}^n e_{m+i}2^{i-1}$ if $(e_n, e_{n-1}, \ldots, e_{m+t+1}) = (0, 0, \ldots, 0)$; $G(\tilde{u}) = \infty$ otherwise.

**Conclusion 1.** The $N, P, T$ classification [1], [2], [3] and the GSG-function of any $\tilde{u} = (a_1, \ldots, a_n)$ can be computed polynomially. In particular, the values $\tilde{Q}(\tilde{u}_i)$, where $\tilde{u}_i = (e_1, \ldots, e_n)$, $e_i = 1, e_j = 0$ $(j \neq i; i = 1, \ldots, n)$, determine $G(\tilde{u})$. Indeed,

\[ Q(\tilde{u}) = \sum_{a_i=1} Q(\tilde{u}_i) = (\delta_n, \delta_{n-1}, \ldots, \delta_{m+1}), \]

and so $G(\tilde{u}) = Q(\tilde{u})$ if $\delta_n = \cdots = \delta_{m+t+1} = 0$; $G(\tilde{u}) = \infty$ otherwise. This, however, does not yet guarantee the realization of a winning strategy, because of possible cycling.

Let $\tilde{u} \in P = C_0(R)$. Then $\tilde{u}$ has a representation $\tilde{u} = (\tilde{y}_1, \ldots, \tilde{y}_k) \subset \mathcal{S}_0(R)$ $(k \leq n)$ in the sense that $\tilde{u} = \Sigma_{i=1}^k \tilde{y}_i$. For example, initially we may have $\tilde{u} \subset \beta_0$. Let $c$ be a monotonic counter function on $C^{(4)}(R)$ (i.e., $G(\tilde{u}) < G(\tilde{v}) \Rightarrow c(\tilde{u}) < c(\tilde{v}))$. We can choose $c(\tilde{u}) = O(n^4)$ for all $\tilde{u} \in V(C^{(4)}(R))$. Let $c(\tilde{u}) = \Sigma_{i=1}^k c(\tilde{y}_i)$. Then $c = O(n^5)$.

**Theorem 2.** There is a function $\Lambda_0$ which can be computed polynomially, such that for every $\tilde{u} \in C_0(R)$ and every $\tilde{v} \in F(\tilde{u})$,
\[ \Lambda_0(\tilde{u}, \tilde{v}) = \tilde{w} = (\bar{w}_1, \ldots, \bar{w}_k) \subset \mathcal{G}_0(R), \]

\[ \bar{w} = \sum_{i=1}^{k} \bar{w}_i \in F(\tilde{v}) \cap P, \]

\[ \tilde{c}(\tilde{w}) < \tilde{c}(\tilde{u}). \]

Note. The representation \( \tilde{w} \) is obtained from \( \tilde{u} \) in a bounded number of transformations. Details are omitted.

Conclusion 2. Using \( \Lambda_0 \) and starting from any \( N \)-position, every annihilation game can be won in \( O(n^5) \) moves using polynomial computation time throughout. The function \( \Lambda_0 \) implies a winning strategy in the wide sense [3]. A bounded number of cycles may be traversed in realizing the strategy (but no cycling takes place in the "representation space"). We do not know if a winning strategy in the narrow sense exists which is always polynomial.

Further results, ramifications and proofs will appear elsewhere.

REFERENCES


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