GENERATORS FOR ALGEBRAS OF RELATIONS

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Let $\mathcal{B}_n$ denote the collection of all binary relations on the set $X = \{1, 2, \ldots, n\}$. The purpose of this paper is to observe that there exists a pair of relations on $X$ that generate all of $\mathcal{B}_n$ under the boolean operations and relational composition.

In [1] C. J. Everett and S. M. Ulam introduced the notion of an abstract projective algebra. McKinsey [2] showed that every projective algebra is isomorphic to a subalgebra of a complete atomic projective algebra and thus, in view of the representation given in [1], every projective algebra is isomorphic to a projective algebra of subsets of a direct product; that is, to an algebra of relations.

**PROJECTIVE ALGEBRA.** A boolean algebra $\mathcal{P}$ with unit 1 and zero 0, so that for all $x \in \mathcal{P}$, $0 \leq x \leq 1$, is said to be a projective algebra if there are defined two mappings $\pi_1$ and $\pi_2$ of $\mathcal{P}$ into $\mathcal{P}$ satisfying the following:

1. $\pi_i(a \lor b) = \pi_i a \lor \pi_i b$.
2. $\pi_1 \pi_2 1 = p_0 = \pi_2 \pi_1 1$ where $p_0$ is an atom of $\mathcal{P}$.
3. $\pi_1 a = 0$ if and only if $a = 0$.
4. $\pi_1 \pi_2 a = \pi_1 a$.
5. For $0 < a \leq \pi_1 1$, $0 < b \leq \pi_2 1$, there exists an element $a \diamond b$ such that $\pi_1 (a \diamond b) = a$, $\pi_2 (a \diamond b) = b$, with the property that $x \in \mathcal{P}$, $\pi_1 x = a$, $\pi_2 x = b$ implies $x \leq a \diamond b$.
6. $\pi_1 1 \cdot a = \pi_1 1; p_0 \cdot \pi_2 1 = \pi_2 1$.
7. $0 < x, y \leq \pi_1 1$ implies $(x \lor y) \cdot \pi_2 1 = (x \cdot \pi_2 1) \lor (y \cdot \pi_2 1)$; and $0 < u, v \leq \pi_2 1$ implies $\pi_1 1 \cdot (u \lor v) = (\pi_1 1 \cdot u) \lor (\pi_1 1 \cdot v)$.

If the projective algebra $\mathcal{P}$ is a complete atomic boolean algebra, then $\mathcal{P}$ is called a *complete atomic projective algebra*. The projective algebra $\mathcal{P}$ is said to be *projectively generated* by a subset $A$ if $\mathcal{P}$ can be obtained from $A$ using $\pi_1$, $\pi_2$, $\cdot$ and the boolean operations.

Consider $\mathcal{B}_n$ and let $p_0 = (1, 1)$. We define the mappings $\pi_1$, $\pi_2$: $\mathcal{B}_n \rightarrow \mathcal{B}_n$ and a product $\cdot$: $\mathcal{B}_n \times \mathcal{B}_n \rightarrow \mathcal{B}_n$ as follows:

(i) $\pi_1 \alpha = \alpha(X \times X)p_0$,
(ii) $\pi_2 \alpha = (p_0(X \times X))\alpha$,
(iii) $\alpha \cdot \beta = (\alpha(X \times X))\beta$. 


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where juxtaposition denotes the *composition* of relations.

It is easy to verify axioms $P_1-P_7$ to establish that $B_n$ with the atom $p_0$ and the mappings $\pi_1, \pi_2$ and $\Box$ as defined is a projective algebra.

The verification above, as well as the calculations below are made easier by noting the following equivalent forms of (i), (ii) and (iii):

(i) $\pi_1 \alpha = \text{domain}(\alpha) \times \{1\}$;
(ii) $\pi_2 \alpha = \{1\} \times \text{range}(\alpha)$;
(iii) $\alpha \Box \beta = \text{domain}(\alpha) \times \text{range}(\beta)$.

**Theorem 1.** The projective algebra $B_n$ can be projectively generated by a pair of disjoint elements.

**Proof.** We observe first that if we generate the atoms $(1, k)$ and $(k, 1)$, $1 \leq k \leq n$, then all others are obtained by taking the $\Box$-product of suitable pairs of these.

Let $\alpha_0 = \{(x, y) | x < y\}$ and $\beta_0 = \{(x, y) | y < x\}$. Now $p_0 = (1, 1) = \pi_2 \pi_1 \beta_0$. If we let $\alpha_i = \alpha_0 - (\pi_0 \pi_2 \alpha_0)$ and $\beta_i = \beta_0 - (\pi_1 \beta_0 \pi_0)$, we get $(\pi_1 \alpha_1 - \pi_1 \beta_1) = (2, 1)$ and $(\pi_2 \beta_1 - \pi_2 \alpha_1) = (1, 2)$. Using the recursions $\alpha_{k+1} = \alpha_k - ((k + 1, 1) \pi \pi_{2k})$ and $\beta_{k+1} = \beta_k - (\pi_1 \beta_k \pi_0 (1, k + 1))$, noting that $\alpha_k = \{(x, y) | k < x < y\}$ and $\beta_k = \{(x, y) | k < y < x\}$, we see that $(\pi_1 \alpha_k - \pi_1 \beta_k) = (k + 1, 1)$ and $(\pi_2 \beta_k - \pi_2 \alpha_k) = (1, k + 1)$, for all $0 \leq k \leq n - 2$. Also $\pi_1 \beta_{n-2} = (n, 1)$ and $\pi_2 \alpha_{n-2} = (1, n)$, so that we have generated all of the atoms mentioned above.

**Theorem 2.** The algebra of relations $B_n$ can be generated, with respect to the boolean operations and composition, by two relations.

**Proof.** Let $\bar{\alpha} = \alpha_0 \cup \{(1, 1)\}$ and $\bar{\beta} = \beta_0 \cup \{(1, 1)\}$. Now $\bar{\alpha} \cap \bar{\beta} = \{(1, 1)\} = p_0$, $\bar{\alpha} \cup \bar{\beta} \cup \bar{\beta} = X \times X$, $\alpha_0 = \bar{\alpha} - p_0$ and $\beta_0 = \bar{\beta} - p_0$. Since we defined the mappings $\pi_1, \pi_2$ and $\Box$ in terms of the composition in $B_n$, Theorem 2 is an immediate consequence of Theorem 1.

**Remark.** It is well-known that $B_n$ cannot be generated by a pair of elements using only the boolean operations. Moreover one can show that the compositional semigroup $B_n$ cannot be generated by a pair of relations.

**References**


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