concerning numbers which are representable as a sum of two squares. Selberg **[14]** has shed further light on the relationship between the large sieve and the Selberg sieve. Diamond and Jurkat (unpublished) have extended the analysis of the iterated Selberg sieve to dimension $\kappa \neq 1$ (see also Porter **[11]**). Bombieri **[2]**, **[3]** has had some innovative ideas concerning weighted sieves. Vaughan **[15]** has given a simple proof of a sharp form of Bombieri's mean value theorem.

For years to come, *Sieve methods* will be vital to those seeking to work in the subject, and also to those seeking to make applications. The heavy notation in the book seems to be essential in formulating such general methods. Some parts of the book are much more difficult to read than others, but generally the text is lively and conversational. In concept and execution this is an excellent, long-needed work.

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AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 6, November 1976

Fourier series with respect to general orthogonal systems, by A. M. Olevskiĭ (translated by B. P. Marshall and H. J. Christoffers), Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 86, Springer-Verlag, Berlin, Heidelberg, New York, 1975, viii + 136 pp., $33.60.

Fourier series–the original Fourier series, that is, the ones using trigonometric functions–were the first series of orthogonal functions. They are either very satisfactory or very unsatisfactory, depending on the point of view. For example, it would be pleasant if the Fourier series of continuous functions converged to the functions that gave rise to them (as Fourier seems to have believed, in common with some present-day physicists); but du Bois-Reymond showed that the Fourier series of continuous functions can in fact diverge at some points. However, Dirichlet showed (in effect) that at least the Fourier series of a function of bounded variation converges to the function (succeeding, as J. L. Walsh once remarked, where Fourier failed, because Dirichlet knew more trigonometry); and Fejér showed that the Fourier series of a continuous function is at least $(C, 1)$ summable, so that at this point the representation problem looked fairly satisfactory. Then Kolmogorov showed that the Fourier series of an integrable function can diverge at every point; but eventually Carleson showed that the Fourier series of an L^2 function (hence of a continuous function) must converge almost everywhere (thus almost justifying Fourier's intuition). However, the convergence properties of Fourier series are strongly tied to the natural ordering of the trigonometric functions: Kolmogorov discovered that the Fourier series of an L^2 function can be rearranged into an almost everywhere divergent series. One can also consider trigonometric series that are not initially known to be Fourier series. Such a series may (for example) converge everywhere, in which case it is the Fourier series of its sum; at the other extreme it may have a sequence of partial sums that converge almost everywhere to any measurable function you like. Thus even trigonometric series are by no means as simple as they appear at first.

Olevskiĭ's book is a survey of the developments of the last 15 years in the theory of general systems of orthogonal functions, along the lines of asking which convergence and divergence properties of trigonometric Fourier series carry over to the more general systems. Are other orthogonal systems as good as the trigonometric system, or better, or worse? To what extent are the special properties of the trigonometric system–such as uniform boundedness, ordering, or completeness–decisive in determining the properties of Fourier series? There are isolated classical results along these lines, but a great deal more has been learned recently, so much, indeed, that although Olevskiĭ surveys the field rather fully, he is able to give proofs only of a rather small number of results that particularly interest him. Perhaps inevitably in an expanding field, it is hard to see any unifying ideas; there is a rather overwhelming array of special results, some of them involving such special terminology that they cannot be appreciated without explanation. Fortunately Olevskiiĭ is rather good at saying, in informal language, what the general idea of each class of results really is; and some of the results are quite striking. The proofs presented in the book are extremely "classical"; they depend, for the most part, on ingenious and difficult constructions. It is rather noticeable that the subject is currently being developed mostly in the Soviet Union and Hungary (with a few notable exceptions). Since the book is primarily a survey, it would be impossible to summarize it in detail without making the review almost as long as the book; I shall, however, try to give an idea of the contents by picking out some representative results. Throughout the book we are working on a finite interval and usually with real-valued functions.

The central theme of the first chapter is the question of what can be said about orthonormal systems $\{\varphi_n(x)\}$ that share with the trigonometric functions the property of being uniformly bounded. The possibility of having a Fourier series that diverges at some points depends on the unboundedness of the Lebesgue functions

$$L_n(x) = \int \left| \sum_{k=1}^{n} \varphi_k(x)\varphi_k(t) \right| dt.$$

It turns out that the unboundedness of the $L_n(x)$ follows merely from the uniform boundedness of the set $\{\varphi_n(x)\}$. Moreover, $L_n(x)$, which is asymptotically $k \log n$ for the trigonometric system, cannot (for a uniformly bounded system) even be $o(\log n)$ on a set of positive measure. In particular, it follows that no uniformly bounded orthonormal system can be a basis in space C; this accounts neatly for the fact that the first orthonormal basis for C, the Haar system, is indeed unbounded. These results, and a number of others, depend on a difficult inequality of the author's for the partial sums of a series $\Sigma c_k \varphi_k(x)$. (Another property of uniformly bounded orthonormal systems, which appeared too late for inclusion in this book, is that there is always a Fourier series that diverges on a set of positive measure: Bochkarev, Mat. Sb. **98** (**140**) (1975), 436–449.)

In Chapter II the problem is, given an orthonormal system $\{\varphi_n(x)\}$, to find conditions on sequences $\{c_n\}$ that will make $\Sigma c_n \varphi_n(x)$ converge almost everywhere. For the trigonometric system we now know (Carleson) that $\{c_n\} \in l^2$ suffices by itself; but it has been known for a long time that there are other orthonormal systems for which an L^2 Fourier series can diverge almost everywhere. It is reasonable to ask, then, for a characterization of multipliers $\omega(n)$ such that $\Sigma c_n \varphi_n(x)$ converges almost everywhere when $\Sigma c_n^2 \omega(n) < \infty$. The boundary between successful and unsuccessful factors is at $\omega(n) = (\log n)^2$ (this is the Rademacher-Menshov theorem); for monotonic $\{c_n\}$ the author presents a tidy recent result by Tandori: if $c_n \downarrow$ then a necessary and sufficient condition for $\Sigma c_n \varphi_n(x)$ to converge almost everywhere for *all* orthonormal systems $\{\varphi_n\}$ is the convergence of $\Sigma c_n^2 (\log n)^2$.

In another direction, it is classical (Paley and Zygmund) that for an arbitrary orthonormal system $\{\varphi_n\}$ almost all series $\Sigma \pm c_n \varphi_n(x)$ converge almost everywhere when $\{c_n\} \in l^2$. The author presents a recent result of Garsia's: the same conclusion holds if, instead of varying the signs of the terms, we vary their order, i.e. almost all rearrangements of $\Sigma c_n \varphi_n(x)$ converge almost everywhere.

If a trigonometric series converges on a set of positive measure its coefficients tend to 0. The property of the trigonometric system that is significant here is the completeness of the system. In fact, for any complete orthonormal system $\{\varphi_n\}$ the convergence of $\Sigma c_n \varphi_n(x)$ on a set of positive measure implies at least that $\Sigma 1/(c_n^2 + 1) = \infty$, and conversely if the latter condition is satisfied, there is a complete orthonormal system $\{\varphi_n\}$ such that $\Sigma c_n \varphi_n(x)$ converges almost everywhere (and also in L^p for $1 \leqslant p < 2$); the novelty is in the second part.

Chapter III considers further properties of complete systems. The central thesis of the chapter is that the Haar system is extremal among complete orthonormal systems: any divergence phenomenon for the Haar system carries over to all complete orthonormal systems. Thus we have, in particular, a new approach to the construction of an L^2 trigonometric series which can be rearranged to diverge almost everywhere (since the construction of such a series of Haar functions is relatively easy). We still do not know whether a trigonometric series can diverge to $+\infty$ on a set of positive measure; this cannot happen for Haar series (Talalyan and Arutyunyan; a neat proof by Skvortsov is given in this chapter). The author also describes a sense in which the Haar system is the best of all bases for the space of continuous functions.

For the trigonometric system we know that the Fourier coefficients of a continuous function can have $\Sigma|c_n|^p = \infty$ for all $p < 2$ (Carleman singularity). This carries over to all complete orthonormal systems, even in a localized sense. Moreover, there is a function, continuous on a compact set K of measure 0, such that every continuous extension from K to the whole interval has a Carleman singularity. Thus the values of a function on a set of measure 0, which do not affect its individual Fourier coefficients, can have a decisive influence on their behavior as a whole. Still another result about the Haar system $\{\chi_n(x)\}$ (Nikishin and and Ulyanov) is that every series $\Sigma c_n \chi_n(x)$ that converges unconditionally almost everywhere also converges absolutely almost everywhere (unconditional convergence almost everywhere means that every rearrangement converges except on a set of measure 0 that may depend on the rearrangement, so that a series that converges unconditionally almost everywhere need not converge absolutely at any point at all).

In Chapter IV we are again concerned with convergence almost everywhere or in L^p, but under conditions on the function instead of on the coefficients. By a classical theorem, the boundedness of the Lebesgue functions $L_n(x)$ at each $x \in E$ makes the Fourier series of every L^2 function converge almost everywhere on E, and uniform boundedness of $L_n(x)$, for a closed system, makes the Fourier series of an L^p function converge in L^p, and also almost everywhere if $p \geqslant 2$. For the usual systems there is also convergence in L^p, $p < 2$; but in the general case, with $L_n(x)$ uniformly bounded, the Fourier series of a function that belongs to every L^p, $p < 2$, can diverge almost everywhere, and for every rearrangement (so that Garsia's theorem does not extend from L^2 to $L^p, p < 2$).

Other theorems in this chapter show, in other ways, how much more complicated than L^2 the L^p spaces for $p < 2$ are. For the trigonometric system, L^p Fourier series converge almost everywhere when $p > 1$, but this is not true for complete orthonormal systems in general. In fact, even for uniformly bounded complete orthonormal systems it can be true, for any $p_0 > 1$, that the Fourier series of functions of L^p converge almost everywhere for $p > p_0$, but can diverge almost everywhere for $p = p_0$. Again, for an L^2 function f, the Fourier series with respect to a complete orthonormal system converges to f in L^2, and so if it converges almost everywhere it converges to f. However, for $p < 2$, even for a complete bounded orthonormal system, the Fourier series of an L^p function can converge to some function other than f; indeed, it is possible to rearrange it (or to take a subsequence of its partial sums) so that it converges to any measurable function we like, or to ∞.

The theory of trigonometric series has given us (for better or worse) many gifts, notably Dirichlet's concept of a function, the Riemann integral and the theory of sets. It remains to be seen whether the general theory of series of orthogonal functions will be as fruitful. Meanwhile we can take comfort from Hermann Weyl's dictum that "special problems in all their complexity constitute the stock and core of mathematics."

The book has not been published in Russian. Unfortunately the translation reads, in uncomfortably many places, like a translation: that is to say, too often for the reader's comfort it preserves Russian word order or sentence structure.

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BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 6, November 1976

Hewitt-Nachbin spaces, by Maurice D. Weir, North-Holland/American Elsevier, Amsterdam, 1975, 270 + vii pp., $15.50.

What does it mean to say that a completely regular, Hausdorff space X is realcompact? To Edwin Hewitt,. who introduced the class of realcompact spaces under the title Q-spaces [**18**], it means that for every maximal ideal M in the ring $C(X, \mathbf{R})$ of continuous real-valued functions on X, either $M = \{f \in C(X): f(p) = 0\}$ for some $p \in X$ or the linearly ordered field $C(X, \mathbf{R})/M$ is non-Archimedean. To a point-set topologist, it means that for some cardinal α the space X is homeomorphic with a closed subspace of the power space $\mathbf{R}^\alpha$ [**34**], [**9**]. To a category-theorist, that X is an object in the epi-reflective hull generated in $\mathfrak{Tych}$ by $\mathbf{R}$ [**16**], [**17**]. To a topological linear space theorist, that $C(X, \mathbf{R})$ is bornological in the compact-open topology [**31**], [**32**], [**35**], or that for every nontrivial multiplicative linear functional Φ on $C(X)$ there is $p \in X$ such that $\Phi(f) = f(p)$ [**18**], [**8**, Problem 3W(b)], [**19**, p. 170]. To a descriptive set theorist, that X is the intersection of Baire subsets of its Stone-Čech compactification βX [**24**, Theorem 9], [**33**, Corollary 3.11]. To a uniform spaceman, that X is complete in the uniformity defined by $C(X)$ [**30**], [**18**, p. 92], [**34**]. And so forth. The ubiquity with which the concept appears, and the elegance of the characterizations available in quite diverse contexts, justify both its introduction into the literature over 25 years ago and the present undertaking of a comprehensive survey.

In his Introduction, Professor Weir sets forth briefly and to good effect the historical data which led him to adopt the name "Hewitt-Nachbin spaces" for the classes he studies here. It is an elementary courtesy due the author that today's remarks in review follow his lead in this respect, but I reserve the