for $p > p_0$, but can diverge almost everywhere for $p = p_0$. Again, for an L^2 function f, the Fourier series with respect to a complete orthonormal system converges to f in L^2, and so if it converges almost everywhere it converges to f. However, for $p < 2$, even for a complete bounded orthonormal system, the Fourier series of an L^p function can converge to some function other than f; indeed, it is possible to rearrange it (or to take a subsequence of its partial sums) so that it converges to any measurable function we like, or to ∞.

The theory of trigonometric series has given us (for better or worse) many gifts, notably Dirichlet's concept of a function, the Riemann integral and the theory of sets. It remains to be seen whether the general theory of series of orthogonal functions will be as fruitful. Meanwhile we can take comfort from Hermann Weyl's dictum that "special problems in all their complexity constitute the stock and core of mathematics."

The book has not been published in Russian. Unfortunately the translation reads, in uncomfortably many places, like a translation: that is to say, too often for the reader's comfort it preserves Russian word order or sentence structure.

R. P. BOAS

BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 6, November 1976

Hewitt-Nachbin spaces, by Maurice D. Weir, North-Holland/American Elsevier, Amsterdam, 1975, 270 + vii pp., $15.50.

What does it mean to say that a completely regular, Hausdorff space X is realcompact? To Edwin Hewitt, who introduced the class of realcompact spaces under the title Q-spaces [**18**], it means that for every maximal ideal M in the ring $C(X, \mathbf{R})$ of continuous real-valued functions on X, either $M = \{f \in C(X): f(p) = 0\}$ for some $p \in X$ or the linearly ordered field $C(X, \mathbf{R})/M$ is non-Archimedean. To a point-set topologist, it means that for some cardinal α the space X is homeomorphic with a closed subspace of the power space $\mathbf{R}^\alpha$ [**34**], [**9**]. To a category-theorist, that X is an object in the epi-reflective hull generated in $\mathfrak{Tych}$ by $\mathbf{R}$ [**16**], [**17**]. To a topological linear space theorist, that $C(X, \mathbf{R})$ is bornological in the compact-open topology [**31**], [**32**], [**35**], or that for every nontrivial multiplicative linear functional Φ on $C(X)$ there is $p \in X$ such that $\Phi(f) = f(p)$ [**18**], [**8**, Problem 3W(b)], [**19**, p. 170]. To a descriptive set theorist, that X is the intersection of Baire subsets of its Stone-Čech compactification βX [**24**, Theorem 9], [**33**, Corollary 3.11]. To a uniform spaceman, that X is complete in the uniformity defined by $C(X)$ [**30**], [**18**, p. 92], [**34**]. And so forth. The ubiquity with which the concept appears, and the elegance of the characterizations available in quite diverse contexts, justify both its introduction into the literature over 25 years ago and the present undertaking of a comprehensive survey.

In his Introduction, Professor Weir sets forth briefly and to good effect the historical data which led him to adopt the name "Hewitt-Nachbin spaces" for the classes he studies here. It is an elementary courtesy due the author that today's remarks in review follow his lead in this respect, but I reserve the right (and announce the intention) to return to former habits tomorrow. Indeed the term "realcompact", proposed (with a hyphen subsequently dropped) by Gillman [12] and adopted in his widely read textbook with Jerison [13], has become generally accepted. Satisfied and content in the familiarity of the status quo we may admire the courage of a crusader; but we applaud only rarely, and we follow more rarely still.

An overview. Given a family $\mathfrak{E}$ of completely regular Hausdorff spaces, the modern point-set topologist is likely to ask the following questions.

(1) Can I define and consider the class $\mathfrak{R}\mathfrak{E}$ of spaces homeomorphic to a closed subspace of a product of elements of $\mathfrak{E}$? Will not $\mathfrak{R}\mathfrak{E}$ be closed under products, and under passage from one of its elements to a closed subspace?

(2) If X is a (not necessarily closed) subset of a product of elements of $\mathfrak{E}$, can I associate with X an "$\mathfrak{E}$-envelope" $\beta_{\mathfrak{E}}X$ in $\mathfrak{R}\mathfrak{E}$ in which X is dense and $\mathfrak{E}$-embedded (in the sense that every continuous function $f: X \to E \in \mathfrak{E}$ extends continuously to $\bar{f}: \beta_{\mathfrak{E}}X \to E$)? If $X \in \mathfrak{E}$, will I not have $\beta_{\mathfrak{E}}X = X$? Is $\beta_{\mathfrak{E}}(X \times Y) = \beta_{\mathfrak{E}}X \times \beta_{\mathfrak{E}}Y$?

(3) Does the class $\mathfrak{R}\mathfrak{E}$ admit alternative characterizations? Are there interesting related classes?

(4) If f is continuous from X_0 onto X_1 and $X_i \in \mathfrak{R}\mathfrak{E}$ $(i = 0, 1)$, is $X_{1-i} \in \mathfrak{R}\mathfrak{E}$?

To oversimplify considerably: Weir's book has four chapters, each devoted to posing (more carefully and more fully than has been done above) and then answering one of the four sets of questions just listed. We comment briefly on these questions and Weir's treatment, beginning at the end and working forward. In what follows it is understood that all spaces considered are completely regular Hausdorff spaces and that $\mathfrak{E} = \{\mathbf{R}\}$, so that $\mathfrak{R}\mathfrak{E}$ is the class of Hewitt-Nachbin spaces. When convenient we write υX in place of $\beta_{\mathfrak{E}}X$.

(4) Since every discrete space whose cardinality is accessible from ω by the usual processes of cardinal arithmetic is a Hewitt-Nachbin space (Mackey [25]), the continuous image of a Hewitt-Nachbin space need not be a Hewitt-Nachbin space. Equally simple examples show that the class is not preserved by inverse images under continuous functions. Thus in seeking conditions under which the image or inverse image of a Hewitt-Nachbin space is again a Hewitt-Nachbin space, it is natural to consider either spaces that are "more than Hewitt-Nachbin" or functions that are "more than continuous". In Chapter 4, a strong and impressive contribution to the literature, Weir has assembled and juxtaposed a tasteful and complete collection of results (due to Blair, Dykes, Frolík, Isiwata, Mack, Mrowka, Tsai and others) concerning preservation and inverse-preservation of many classes of spaces of Hewitt-Nachbin type. This chapter is carefully organized and comprehensive, and it serves a function not served (so far as I am aware) by any competing survey of the subject.

(3) Several alternative definitions, or characterizations, of Hewitt-Nachbin spaces are given in the first paragraph of this review.

(2), (1) It was Tychonoff [38] who showed that a (completely regular, Hausdorff) space X can be embedded in $[0, 1]^{\alpha}$, where α is the weight of X.

Čech [3] in effect credits Tychonoff with the statement that if $\mathfrak{F} = C(X, [0, 1])$ then the function $e: X \to P = [0, 1]^{\mathfrak{F}}$ defined by $(ex)_f = f(x)$ is a topological embedding, and Čech shows that eX is $[0, 1]$-embedded in $\mathrm{cl}_P eX$. This space, which he denoted βX, is the Stone-Čech compactification; Stone's construction [37], given simultaneously by a quite different method, should not concern us here.

The Tychonoff-Čech device of embedding in a product in such a way that functions on the image extend (because they are essentially projections) has been used frequently since 1937. Hewitt [18] showed his Q-spaces to be homeomorphic with closed subspaces of the products $\mathbf{R}^\alpha$, and he constructed υX by analogy with βX.[1] The proof of the adjoint functor theorem of Freyd [10], [11] achieves similar results in a setting of greater (indeed, maximal) generality; see also Kan [21], Bénabou [2], Mitchell [26] and Lawvere [23]. Better known to topologists are the more recent papers of Kennison [22], van der Slot [36], Herrlich [15] and Herrlich and van der Slot [17], in which (collectively) the adjoint functor theorem and its consequences are set forth in sufficient generality to handle any question likely to arouse the interest of the point-set topologist. Probably the earliest paper of this sort in general topology–containing (in different language) essentially all one needs to know about simply generated epi-reflective subcategories of $\mathfrak{Tych}$–is Engelking and Mrowka [9]. This paper, which appeared just too late to have a substantial influence on [13], is a primary source for Weir.

An overly enthusiastic view of the facts that the product of compact spaces is compact and the product of Hewitt-Nachbin spaces is a Hewitt-Nachbin space can lead to the belief that $\beta(X \times Y) = \beta X \times \beta Y$ and $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ for all spaces X and Y. The following pretty theorem of Glicksberg [14], whose proof is not included in the present work, shows that the former equality (which implies the latter) occurs infrequently: For infinite spaces X and Y, $\beta(X \times Y) = \beta X \times \beta Y$ if and only if $X \times Y$ is pseudocompact. Since the publication of Weir's book, Hušek has verified a suspicion articulated tentatively but inadequately over the years by various investigators admiring Glicksberg's result: there is no similarly elegant solution to the υ problem. Specifically, Hušek's beautiful paper [20] shows that if $\mathfrak{K}$ is a finitely productive class of Tychonoff spaces containing compacta and containing a pair $\langle P, Q \rangle$ of spaces such that $\upsilon(P \times Q) \neq \upsilon P \times \upsilon Q$, then there are no reflective subcategories $\mathfrak{R}$, $\mathfrak{S}$ of $\mathfrak{Tych}$ with reflectors r, s such that for every $X, Y \in \mathfrak{K}$ we have: $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ if and only if $r(X \times Y) = s(X \times Y)$.

[1] I was informed by a letter from Professor Hewitt dated 24 May, 1966 that he "chose upsilon by some crude association with the word "unbounded", just as Čech probably chose β because he was thinking of bounded functions". The symbol υ, though frequently miswritten ν (nu) by Greekless authors and typesetters (for an early example of this phenomenon see Hewitt [19, p. 178]), is by now time-tested and indeed universally accepted: clearly an excellent choice of notation. But at the risk of taking unfair advantage of Professor Hewitt, who does not have the opportunity to respond on this page, I respectfully take issue with his analysis of Čech's motivation. In the absence of compelling and specific information to the contrary, I suspect that Čech chose the symbol β for its affinity with the word "bicompact", or in order to contrast with the notation αX (used commonly, when X is locally compact but not compact, to denote the one-point compactification of Alexandroff (cf. [1])). Indeed it seems likely that if Čech had been led to consider with emphasis the *bounded* real-valued continuous functions on X, he might have continued next to the unbounded ones, thus perhaps constructing on his own an early version of Hewitt's space υX.

Weir's book, like the successful and widely consulted textbook [39] of his Carnegie-Mellon graduate-school colleague Russell C. Walker, is derived from a Doctor of Arts thesis presented to that university. Together the two books constitute proof of the integrity and worth of that particular program and, more generally, they testify that this much-maligned degree is potentially beneficial not only to individual recipients but also to a broad reading audience. Nevertheless a manuscript highly suitable as a graduate thesis may fail to meet some of the standards of accuracy and maturity normally associated with formal publication. Perhaps it is less appropriate to censure Weir for the several minor errors in fact or in judgement discussed in the list below than it is to congratulate him that the list is so short.

(a) A perplexing misprint occurs on p. 68, where it is asserted that the space $[0, \Omega]$ of ordinals less than or equal to the first uncountable ordinal is not a Hewitt-Nachbin space. In fact this space, like any well-ordered space with a greatest element, is compact (and *a fortiori* a Hewitt-Nachbin space). Presumably reference is intended to the space $[0, \Omega)$ of countable ordinals. The difficulty is compounded by the unfortunate definition

$$(\alpha, \beta) = \{x: x > \alpha\} \cap \{x: x < \beta + 1\},$$

given on the same page.

(b) The space $[0, \Omega)$ is again subjected to rough handling (p. 152) in connection with the statement that its topology admits no countable basis. To go around by way of paracompactness and a theorem which discusses measurable cardinals, as Weir does, is just too devious: the space $[0, \Omega)$ is, quite clearly, not separable, so it has no countable basis.

(c) On p. 87, the assertion is made that a certain hypothesis in Corollary 8.11 is so essential that it cannot be dropped. In fact, however, it is so inessential that Weir himself has dropped it; it does not appear.

(d) The pattern of proof of the equivalences in Theorem 8.4 is as follows: $(1) \Rightarrow (2)$, $(2) \Rightarrow (4)$, $(4) \Rightarrow (3)$, $(3) \Rightarrow (4)$, $(3) \Rightarrow (5)$, and $(5) \Rightarrow (1)$. There is a redundancy here.

(e) The following sentence, quoted from p. 92, is brief and incomplete to the point of irresponsibility: "A celebrated unsolved problem is whether or not <u>every</u> cardinal is nonmeasurable." Interpreted kindly, the statement is probably correct: it is conceivable that someday someone will prove that there are no measurable cardinals.[2] Contrary to the impression one gains from Weir's statement, however, a little something is known. Specifically, it is consistent with ZFC that every (infinite) cardinal is nonmeasurable: that is, the existence of a measurable cardinal cannot be proved in ZFC. Further, it

[2] Out of respect for Weir as author, we here use the term "measurable" exactly as he does: A cardinal number α is measurable if there is on α a nonprincipal ultrafilter with the countable intersection property. According to this terminology, which is used also in [13], the cardinal ω is not measurable, and any cardinal exceeding the least measurable cardinal is itself measurable. Many current authors, however, call cardinals with the property just described Ulam-measurable, reserving the term measurable for those cardinals α on which there is a nonprincipal α-complete ultrafilter (*i.e.*, an ultrafilter p such that $\bigcap \mathcal{F} \in p$ whenever $\mathcal{F} \subset p$ and $|\mathcal{F}| < \alpha$). According to this terminology (and assuming, as is usual, the Axiom of Choice), the cardinal ω is measurable and no successor cardinal is measurable. For basic facts and references concerning both kinds of cardinals, see for example [5, especially pp. 186–203] or [7].

follows from the Incompleteness Theorem of Gödel (see for example Cohen [**4**, p. 45] or Crossley [**6**]) that even the consistency with ZFC of the statement that there is an infinite measurable cardinal cannot be shown in ZFC. In any event, most set theorists today are willing to assume the existence of large cardinal numbers with various exotic properties, including the property of measurability.

(f) An interesting and crucial example of Mrowka [**27**], [**28**] is discussed (pp. 189–191): The image of a Hewitt-Nachbin space under a perfect function need not be a Hewitt-Nachbin space. (A function f from X onto Y is said to be perfect if f is a closed, continuous function such that $f^{-1}(\{y\})$ is compact for each $y \in Y$.) The construction of Mrowka's example and the verification of its properties is, shall we say, a 15-step procedure. It is annoying and aesthetically unsatisfactory that Weir offers the details of steps 1–6 and 8–15, but passes over step 7 with the statement "Mrowka then proves, using an additional lemma, that there exists a permutation π . . . ". A chain with a missing link is no chain at all; if a complete proof of Mrowka's result is unsuitable to the text, I should have preferred either a brief, informal description of the space in question or simply a reference to [**27**], [**28**].

(g) To ensure that Corollary 16.14 has content, it seems necessary to show by an example that the pre-image X under a perfect function f of a pseudocompact space Y need not be pseudocompact. An example can be given whenever Y is a pseudocompact space with a closed, nonpseudocompact subset A (such pairs are easily found). Let X be the "disjoint union" $(A \times \{0\}) \cup (Y \times \{1\})$, and set $f(a, 0) = a, f(y, 1) = y$.

(h) On p. 29 we are told "Mrowka has shown . . . that strong zero-dimensionality is preserved under products, but we omit that argument here". In fact the question is raised but not settled in the indicated paper of Mrowka [**29**], and it remains open still.

(i) It is surely reasonable and honorable, when confronted with so careful and elegant a text as the book by Gillman and Jerison [**13**], to quote generously from it and to refer the reader frequently to it. Weir does this freely and openly. Through carelessness, however, he gives the impression that many of the results quoted from [**13**] (*e.g.*, Urysohn's Lemma, several results essentially contained in Hewitt's paper [**18**]) originated in [**13**]. To avoid embarrassment all around, one wishes that Weir had developed two sorts of notation to be used when citing results from other sources–one to indicate simply that Weir had found the result there, the other to suggest where the result originated.

The role of the bibliography, which contains 327 items, is not clear. Many of the items listed are not mentioned or cited in the text, and some even appear to be effectively disjoint from it.

Finally, a few truly petty questions and criticisms. (1) There is a tendency to undefined, unnecessary adjectives: "A normal base on X is a distinguished collection [of subsets of X]" such that . . . (p. 57); "Let R be an algebraic ring with identity" (p. 59). The fully educated reader will know, instinctively or through experience, that "distinguished" and "algebraic" are, in these contexts, meaningless words introduced only in the interest of poetic rhythm; but the ignorant reader (*e.g.*, this reviewer) will waste time thumbing, first backward into the text, next forward into the index, looking for the technical definitions. (2) To help his reader, Weir notes (p. 49) that "There are several good references for the usual concept of a filter such as the 1966 English version of N. Bourbaki . . . ". Are we to infer that the concept is not well handled in the earlier French version? (3) Theorem 4.2 is attributed to Mrowka and Engelking, 4.3 to Engelking and Mrowka. If there is a story behind the distinction, I should like to know it.

In summary. The class of Hewitt-Nachbin spaces has won the respectful interest of many point-set topologists in the past quarter-century, and the spaces themselves have served to clarify and to solve problems in a variety of mathematical disciplines. Professor Weir and his mathematical advisors correctly defined and identified a substantial topic ripe for summary in monograph form, and the present volume responds usefully to a gap in the literature.

Whether this is the definitive treatment of the subject is difficult to guess. Čech's β and Hewitt's v are now routinely perceived as specific instances of a categorical construction widely applicable, and it seems likely that they will in the future recede to the role of interesting illustrative examples. If this occurs there will be no demand for a more complete and coherently crafted treatise which accomplishes for $C(X)$ and vX all that Gillman and Jerison **[13]** and Walker **[39]** have achieved for $C^*(X)$ and βX, and Weir's work will stand as the last and the best word available. In any event the international topological community owes Weir a vote of thanks: He has demonstrated the utility and the wide applicability of this concept in language susceptible to study by the mathematical layman and he has recorded sufficiently many deep and technical theorems and proofs to make his book a valuable–indeed, indispensable–companion to the active researcher.

REFERENCES

1. P. Alexandroff and P. Urysohn, *Mémoire sur les espaces topologiques compacts*, Verh. Kon. Akad. Vetensch. Amsterdam **14** (1929), 1–96.

2. Jean Bénabou, *Criteres de représentabilité des foncteurs*, C. R. Acad. Sci. Paris **260** (1965), 752–755. MR **31** #222.

3. Eduard Čech, *On bicompact spaces*, Ann. of Math. (2) **38** (1937), 823–844.

4. Paul J. Cohen, *Set theory and the continuum hypothesis*, Benjamin, New York and Amsterdam, 1966. MR **38** #999.

5. W. W. Comfort and S. Negrepontis, *The theory of ultrafilters*, Grundlehren math. Wiss., Band 211, Springer-Verlag, Berlin and New York, 1974.

6. J. N. Crossley, *What is mathematical logic*?, Oxford Univ. Press, London and New York, 1972.

7. Frank R. Drake, *Set theory*: *An introduction to large cardinals*, North-Holland, Amsterdam, 1974.

8. R. Engelking, *Outline of general topology*, PWN, Warsaw, 1965; English transl., North-Holland, Amsterdam; PWN, Warsaw; Interscience, New York, 1968. MR **36** #4508; **37** #5836.

9. R. Engelking and S. Mrówka, *On E-compact spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **6** (1958), 429–436. MR **20** #3522.

10. Peter J. Freyd, *Functor theory*, Ph. D. Dissertation, Princeton Univ., 1960.

11. ———, *Abelian categories*: *An introduction to the theory of functors*, Harper and Row, New York, 1964. MR **29** #3517.

12. Leonard Gillman, *Real-compact spaces* (*Q-spaces*), Bull. Amer. Math. Soc. **63** (1957), 144–145 (Abstract).

13. Leonard Gillman and Meyer Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, N. J., 1960. MR **22** #6994.

14. Irving Glicksberg, *Stone-Čech compactifications of products*, Trans. Amer. Math. Soc. **90** (1959), 369–382. MR **21** #4405.

15. Horst Herrlich, *𝔈-kompakte Räume*, Math. Z. **96** (1967), 228–255. MR **34** #5051.

16. ———, *On the concept of reflections in general topology*, Contributions to Extension Theory of Topological Structures (Proc. Sympos., Berlin, 1967), VEB Deutsch. Verlag Wissensch., Berlin, 1969, pp. 105–114. MR **44** #2210.

17. H. Herrlich and J. van der Slot, *Properties which are closely related to compactness*, Nederl. Akad. Wetensch. Proc. Ser. A **70** = Indag. Math. **29** (1967), 524–529. MR **36** #5898.

18. Edwin Hewitt, *Rings of real-valued continuous functions* I, Trans. Amer. Math. Soc. **64** (1948), 45–99. MR **10**, 126.

19. ———, *Linear functionals on spaces of continuous functions*, Fund. Math. **37** (1950), 161–189. MR **13**, 147.

20. M. Hušek, *Hewitt realcompactification of products*, Topics in Topology (Proc. Colloq., Keszthely, 1972), Colloq. Math. Soc. János Bolyai, Vol. 8, North-Holland, Amsterdam, 1974, pp. 427–435. MR **51** #9029.

21. Daniel M. Kan, *Adjoint functors*, Trans. Amer. Math. Soc. **87** (1958), 294–329. MR **24** #A1301.

22. J. F. Kennison, *Reflective functors in general topology and elsewhere*, Trans. Amer. Math. Soc. **118** (1965), 303–315. MR **30** #4812.

23. F. William Lawvere, *Functorial semantics of algebraic theories* (*with appendix*), Ph. D. Dissertation, Columbia Univ., 1963.

24. E. R. Lorch, *Compactification, Baire functions and Daniell integration*, Acta Sci. Math. (Szeged) **24** (1963), 204–218. MR **29** #584.

25. George W. Mackey, *Equivalence of a problem in measure theory to a problem in the theory of vector lattices*, Bull. Amer. Math. Soc. **50** (1944), 719–722. MR **6**, 70.

26. Barry Mitchell, *Theory of categories*, Academic Press, New York and London, 1965. MR **34** #2647.

27. S. Mrówka, *On the unions of Q-spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **6** (1958), 365–368. MR **21** #1572.

28. ———, *Some comments on the author's example of a non-$\mathfrak{R}$-compact space*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **18** (1970), 443–448. MR **42** #3749.

29. ———, *Recent results on E-compact spaces and structures of continuous functions*, Proc. Univ. Oklahoma Topology Conf. (1972), Univ. of Oklahoma, Norman, 1972, pp. 168–221. MR **50** #11152.

30. Léopoldo Nachbin, Unpublished notes.

31. ———, *On the continuity of positive linear transformations*, Proc. 1950 Internat. Congress of Mathematicians, Vol. I, Amer. Math. Soc., Providence, R. I., 1952, pp. 464–465.

32. ———, *Topological vector spaces of continuous functions*, Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 471–474. MR **16**, 156.

33. S. Negrepontis, *Baire sets in topological spaces*, Arch. Math. (Basel) **18** (1967), 603–608. MR **36** #3314.

34. Taira Shirota, *A class of topological spaces*, Osaka Math. J. **4** (1952), 23–40. MR **14**, 395.

35. ———, *On locally convex vector spaces of continuous functions*, Proc. Japan Acad. (= Proc. Imperial Acad., Tokyo) **30** (1954), 294–298. MR **16**, 275.

36. J. van der Slot, *Universal topological properties*, Math. Centrum Amsterdam Afd. Zuivere Wisk. 1966, ZW-011. MR **39** #3457.

37. M. H. Stone, *Applications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc. **41** (1937), 375–481.

38. A. Tychonoff, *Über die topologische Erweiterung von Räumen*, Math. Ann. **102** (1930), 544–561.

39. Russell C. Walker, *The Stone-Čech compactification*, Ergebnisse math. Grenzgebiete, vol. 83, Springer-Verlag, New York and Berlin, 1974.

W. W. COMFORT