PERTURBATION AND ANALYTIC CONTINUATION
OF GROUP REPRESENTATIONS

BY PALLE T. JØRGENSEN

Communicated by C. Davis, July 6, 1976

ABSTRACT. I introduce a theory of noncommutative bounded perturbations of Lie algebras of unbounded operators. When applied to group representations, it leads to an analytic embedding of the dual object of some semi-simple Lie groups into the bounded operators on corresponding Hilbert spaces of $K$-finite vectors.

1. Introduction. I announce a general theorem on analytic continuation of group representations which is based on perturbation theory for linear operators. This result is a contribution of the author to a series of joint results with R. T. Moore reported in detail in [3]. Applications of the theorem to quasi-simple Banach representations of $SL(2, \mathbb{R})$, due to Moore, will be announced separately by him. The theorem introduces a perturbation theory for representations of Lie groups which generalizes the classical perturbation theory (due to R. S. Phillips [2, p. 389]) for one-parameter (semi) groups of bounded linear operators on a Banach space. Let $\{\pi(t): -\infty < t < \infty\}$ be such a strongly continuous one-parameter group ($C_0$ group) acting on a Banach space $E$. Let $A$ be the infinitesimal generator of $\pi$, and let $U$ be a "small" (bounded, say) perturbation of $A$, $B = A + U$. Then $B$ generates a $C_0$ group $\{\pi(t)\}$ on $E$, and this group depends analytically on $U$ (in a sense which is specified in [2, p. 404]). In my theorem the real line $\mathbb{R}$ is replaced by a Lie group $G$, and $A$ is replaced by a Lie algebra $L$ of unbounded operators in $E$. $U$ is going to be a tuple $(U_1, \ldots, U_r)$ of bounded operators. In that way I obtain a surprisingly simple analytic continuation picture for a wide class of induced representations, and other unitary and nonunitary representations.

2. Assumptions. I first restrict the class of perturbations $U$ to be considered. In order to make sure that $\pi_U$ is a representation of the same group for all $U$, I assume that the corresponding infinitesimal operator Lie algebras $L_U$ are all algebraically isomorphic.

Let $D$ be a linear space. Let $\mathfrak{U}(D)$ be the algebra of linear endomorphisms of $D$. It is also a real Lie algebra when equipped with the commutator bracket, $[A, B] = AB - BA$ for $A, B \in \mathfrak{U}(D)$. The Lie algebra $L$ generated by a subset $S$ of $\mathfrak{U}(D)$ is defined to be the smallest real Lie subalgebra of $\mathfrak{U}(D)$ which contains


1Sponsored by Odense University, Denmark.
Let $L_0$ be a finite-dimensional Lie subalgebra of $\mathfrak{U}(D)$, and let $A = (A_1, \ldots, A_r)$ be a basis for $L_0$. The order of the operators $A_k$ is essential only when addition of operator tuples is performed: $A + U = B$ with $B_k = A_k + U_k$ and $U_k \in \mathfrak{U}(D)$ for $1 \leq k \leq r$. I consider $r$-tuples $U$ with the property that the Lie algebra $L_U$ generated by $A + U$ is algebraically isomorphic to some fixed finite-dimensional Lie algebra $\mathfrak{g}$ for all $U$: $L_U \approx \mathfrak{g}$. The set of such $r$-tuples is denoted by $U$. Then there are simply connected Lie groups $G$ (resp. $G_0$) with Lie algebras $\mathfrak{g}$ (resp. $\mathfrak{g}_0$) such that $L_0 \approx \mathfrak{g}_0$.

I restrict the class $U$ of perturbations further. Let $\| \cdot \|$ be a fixed norm on $D$. Put $\| x \|_0 = \| x \|$ and $\| x \|_n = \max \{ \| A_{i_1} \cdots A_{i_n} x \| : 0 \leq i_k \leq r \}$ for $x \in D$ and $n = 1, 2, \ldots$. (Define $A_0$ to be the identity $I$ on $D$.) An element $V \in \mathfrak{U}(D)$ is said to be $\| \cdot \|_n$-bounded if there is a finite constant $c_n$ such that $\| Vx \|_n \leq c_n \| x \|_n$ for all $x \in D$. Let $D_n$ be the completion of $(D, \| \cdot \|_n)$ for $n = 0, 1, \ldots$. Put $D_0 = E$. I assume that the operators $A_k$ are closable when viewed as unbounded operators in $E$. Hence $D_1 \subset E$ (cf. [3]). If $V$ is $\| \cdot \|_0$-bounded, it extends to a bounded operator on $E$, $\overline{V} \in L(E)$. Let $V \in \mathfrak{U}(D)$ be $\| \cdot \|_0$-bounded. Then $V$ is $\| \cdot \|_n$-bounded for given $n$ iff the commutators $[A_{i_1}, [A_{i_2}, \ldots, [A_{i_n}, V] \ldots]]$ are $\| \cdot \|_0$-bounded for all $i$ ($0 \leq i_k \leq r$).

Consider the following subset $U$ of $U$: $U = (U_1, \ldots, U_r)$ belongs to $U$ if and only if each $U_k$ is $\| \cdot \|_0$-bounded and one of the following two conditions is satisfied:

(i) each $U_k$ is $\| \cdot \|_n$-bounded for all $n$; or

(ii) $\{ A_k + U_k \}$ is a basis for $L_U$ and each $U_k$ is $\| \cdot \|_1$-bounded.

3. THEOREM. Let $L_0 \subset \mathfrak{U}(D)$ be an operator Lie algebra, $A = (A_1, \ldots, A_r)$ a basis for $L_0$, and let the class $U$ of bounded perturbations be as described above. Suppose that $L_0$ exponentiates to a $C_0$ representation $\pi$ of $G_0$ on $E$.

(a) Then $L_U$ exponentiates to a $C_0$ representation of $G$ on $E$ for all $U \in U$. We denote the exponential by $\pi_U$.

(b) Let $\Omega$ be a complex domain (in one or several dimensions). Let $z \rightarrow U(z) = (U_1(z), \ldots, U_r(z))$ be an analytic function which is defined on $\Omega$ and has its range in $U$. Then $\pi_U(z)$ is analytic as a function of $z$, i.e., $z \rightarrow \pi_U(z)(g)$ is analytic for all $g \in G$.

(c) The representations $\pi$ and $\pi_U$ have the same space of $C^\infty$-vectors for all $U \in U$.

REMARK. A suitable class of analytic perturbations $U$ gives representations $\pi_U$ which have the same space of analytic vectors as $\pi$.

The proof is based on two exponentiation theorems due to the co-authors of
I state those theorems as lemmas here. They are significant improvements of results announced in [4], and appear below for the first time in their strengthened form.

**Lemma 1.** Let $D$ be a normed linear space, and $E$ the corresponding completion. Let $L \subset \mathcal{U}(D)$ be a finite-dimensional Lie algebra. Suppose $L$ is generated (in the Lie sense) by a subset $S$ such that every $A \in S$ is closable and the closure $\overline{A}$ generates a $C_0$ group $\{\pi(t, A) : t \in \mathbb{R}\} \subset L(E)$.

If $D$ is invariant under $\pi(t, A)$ for $t \in \mathbb{R}$ and $A \in S$, and $t \mapsto \|B\pi(t, A)x\|$ is locally bounded for all $B$, $A \in S$ and $x \in D$, then $L$ exponentiates.

**Lemma 2.** Let $L$ and $S$ be as above. (This means that we have $C_0$ groups $\{\pi(t, A)\}$ for $A \in S$, and there are finite constants $\omega_A$ such that

$$\sup_t e^{-|t|\omega_A} \|\pi(t, A)\| < \infty.$$}

Let $B_1, \ldots, B_d$ be a basis for $L$. Put $B_0 = I$, and

$$\|x\|_1 = \max\{\|B_i x\| : 0 \leq i \leq d\}$$

for $x \in D$.

Suppose each $A \in S$ satisfies the condition: (GD) There are complex numbers $\lambda_{\pm}$ such that $\text{Re } \lambda_+ > \omega_A + |\text{ad } A|$, $\text{Re } \lambda_- < -\omega_A - |\text{ad } A|$, and the ranges of $\lambda_{\pm} I - A$ are $\|\cdot\|_1$-dense in $D$. Then $L$ exponentiates.

**Proof Sketch (a).** Suppose $L_0$ exponentiates to a representation $\pi$. Let $U \in V$, and suppose that (i) holds. Then one may apply bounded Phillips perturbations to each of the spaces $D_n = D_n(\pi)$ (cf. [1, Proposition 1.1]) and conclude that each $\overline{B_k} = \overline{A_k} + \overline{U_k}$ generates a $C_0$ group $\pi(t, B_k)$ which leaves $D_\infty$ invariant. So Lemma 1 applies to $L_U$, with $D$ replaced by $D_\infty$.

If (ii) holds, then apply Lemma 2 to $L_U$. Bounded Phillips perturbation in $D_1$ shows that $\pi(t, B_k)$ restricts to a $C_0$ group in $L(D_1)$. Condition (GD) is a simple consequence of this.

**Remark.** The lemmas are hard to apply directly to operator Lie algebras that arise in applications. Fortunately many of these can be shown to be perturbations of a base-point Lie algebra to which the lemmas easily apply.

At this point I have verified, using the theorem, that the dual $\hat{G}$ of the 3- or the 15-dimensional conformal group is analytically embedded via $\pi_U \to U$ in $B(H)$ for a common Hilbert space $H$. The range consists of operators which are linear combinations of bounded shifts modulo the compacts (and occasionally Hilbert-Schmidt). This gives new and simple metrics on $\hat{G}$, and thus realizes ideas that were recently suggested to me by Professor I. E. Segal.
REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PENNSYLVANIA, PHILADELPHIA, PENNSYLVANIA 19174