FAILURE OF A QUADRATIC ANALOGUE
OF SERRE'S CONJECTURE

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Communicated by Olga Taussky Todd, July 30, 1976

Let \( A \) be a commutative ring with identity. By an inner product \( A \)-space we shall understand, as in [6], a pair \((P, q)\), where \( P \) is a finitely generated projective \( A \)-module and \( q \) is a symmetric bilinear form \( P \times P \rightarrow A \) which is nonsingular (i.e. induces an isomorphism \( P \cong P^* \)). If \( B \) is a commutative \( A \)-algebra we obtain an inner product \( B \)-space \((B \otimes_A P, B \otimes_A q)\). Inner product \( B \)-spaces isomorphic to one of these will be said to be extended from \( A \).

The quadratic analogue of Serre's conjecture is the affirmation of:

Suppose \( A \) is a polynomial algebra \( K[X_1, \ldots, X_n] \) over a field \( K \). Is every inner product \( A \)-space extended from \( K \)?

This question is motivated by the following evidence.

(1) Serre's conjecture that projective \( A \)-modules are free, hence extended from \( K \), has recently been proved by Quillen and Suslin (cf. [4]). Moreover this immediately implies that "symplectic \( A \)-spaces" are extended from \( K \) (see e.g. [1, Chapter IV, (4.11.2)]).

(2) If \( \text{Char}(K) \neq 2 \) then a theorem of Karoubi [7, Theorem 1.1] implies that every inner product \( A \)-space is stably isomorphic to one extended from \( K \).

(3) A theorem of Harder (see [8, Theorem 13.4.3]) gives an affirmative response to (QS) for \( n = 1 \).

A major tool in Quillen's proof of Serre's conjecture is:

QUILLEN'S LOCALIZATION THEOREM [11]. Let \( A \) be a commutative ring, let \( T \) be an indeterminate, and let \( M \) be a finitely presented \( A[T] \)-module. If, for all maximal ideals \( m \) of \( A \), \( M_m \) is extended from \( A_m \), then \( M \) is extended from \( A \).

(4) The analogue of Quillen's localization theorem for inner product spaces has been proved in [3].

The other main tool Quillen uses is:

HORROCK'S THEOREM [5]. Let \( A \) be a local ring and let \( P \) be a finitely generated projective \( A[T] \)-module. If \( P \) extends to a locally free sheaf on \( \mathbb{P}^1_A \), then \( P \) is extended from \( A \) (hence free).
It is natural then to ask:

(QH) *Is the analogue of Horrocks's theorem for inner product spaces valid?*

We here answer (QS) and (QH) negatively, with the following example. Let $A = \mathbb{R}[X, Y]$. Consider the symmetric $4 \times 4$ matrix over

$$A, S = \begin{pmatrix} \alpha & \beta \\ t\beta & \alpha \end{pmatrix}$$

where

$$\alpha = \begin{pmatrix} 4 + Y^2(1 + X^2) & XY(1 + Y^2) \\ XY(1 + Y^2) & 1 + X^2 Y^4 \end{pmatrix} = t\alpha,$$

($t$ denotes transpose), and

$$\beta = \begin{pmatrix} 0 & Y(1 + X^2 Y^2) \\ -Y(1 + X^2 Y^2) & 0 \end{pmatrix} = -t\beta.$$

Let $q$ be the bilinear form on $P = A^4$ with matrix $S$ relative to the natural basis of $A^4$.

**Theorem.** (1) $(P, q)$ is an inner product space over $A = \mathbb{R}[X, Y]$ which is not extended from $\mathbb{R}$.

(2) For each prime ideal $\mathfrak{p}$ of $A$, the inner product $A_\mathfrak{p}$-space $(P_\mathfrak{p}, q_\mathfrak{p})$ is extended from $\mathbb{R}$.

(3) $(P, q)$ extends to a sheaf of inner product spaces over $\mathbb{P}^1_{\mathbb{R}[X]}$, yet for some prime ideal $\mathfrak{p}$ of $\mathbb{R}[X]$, the inner product $\mathbb{R}[X]_\mathfrak{p}[Y]$-space $(P_\mathfrak{p}, q_\mathfrak{p})$ is not extended from $\mathbb{R}[X]_\mathfrak{p}$.

**Remarks.** (a) The matrix $S$ is derived from the hermitian matrix $H = \alpha + i\beta$ over $\mathbb{C}[X, Y]$, which was discovered from an investigation of the classification of the rank 1 projective $\mathbb{H}[X, Y]$-modules ($\mathbb{H}$ = quaternions) in terms of hermitian matrices, established in [9], [10]. The analogue of the above theorem for the hermitian $\mathbb{C}[X, Y]$-space defined by $H$ is also valid.

(b) The matrix $S$ has entries in $\mathbb{Z}[X, Y]$, and $\det(S) = 16$. Thus $(P, q)$ is extended from an inner product space over $\mathbb{Z}[1/2][X, Y]$.

(c) If one considers quadratic forms rather than symmetric bilinear forms the analogue of (QS) has a negative response (in characteristic 2 of course) already for $n = 1$ (see [7, p. 318]).

(d) Bass [2] has investigated (QS) when $K$ is algebraically closed.

(e) It follows from Harder's theorem (3) that the answer to (QH) is trivially in the affirmative if $A$ is a field. Our theorem shows that the answer is negative already for the discrete valuation ring $\mathbb{R}[X]_\mathfrak{p}$.

(f) The proof of the above theorem will appear elsewhere.
I am thankful to Professor H. Bass for carefully going through my calculations and his great help in preparing this manuscript.

I am more than grateful to Professor R. Sridharan to whom this work is respectfully dedicated.

REFERENCES


2. ———, Quadratic modules over polynomial rings (to appear).


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