We recall the notation and results of [3].

Let \( \mathbb{Q} \) be the rational numbers.

We let \( L \) be a \( \mathbb{Q} \) integral lattice in \( \mathbb{Q}^k \), i.e. \( \mathbb{Q}(\xi_1, \xi_2) \in \mathbb{Z} \) for all \( \xi_1, \xi_2 \in L \). Let \( L_\bullet(\mathbb{Q}) \) be the \( \mathbb{Q} \) dual of \( L \), i.e. \( L_\bullet(\mathbb{Q}) = \{ \eta \in \mathbb{R}^k \mid Q(\eta, \xi) \in \mathbb{Z}, \forall \xi \in L \} \).

Then \( L_\bullet(\mathbb{Q})/L \) is a finite Abelian group, and we let \( N_L \) be the exponent of \( L_\bullet(\mathbb{Q})/L \), i.e. the smallest positive integer \( x \) so that \( x \cdot \xi \in \mathbb{Z} \) for all \( \xi \in L_\bullet(\mathbb{Q}) \). Choosing a \( \mathbb{Z} \)-basis \( X_i \) of \( L \), we let \( D_Q(L) = \det(Q(X_i, X_j)) \). Then the integer \( D_Q(L) \) is independent of the choice of basis of \( L \).

Then we define

\[
\Gamma_L(\mathbb{Q}) = \{ g \in O(\mathbb{Q}) \mid g(L) = L \}
\]

and

\[
\Gamma^L(\mathbb{Q}) = \left\{ \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right], \epsilon \mid a, b, c, d \in \mathbb{Z}, \; ad - bc = 1, \right. \left. b \equiv 0 \mod 2 \text{ and } c \equiv 0 \mod 2N_L \right\}.
\]

Then \( \Gamma_L(\mathbb{Q}) \) is an arithmetic subgroup of \( O(\mathbb{Q}) \) and \( \Gamma^L(\mathbb{Q})/(\text{cyclic group of order 4}) \) is an arithmetic subgroup of \( PSL_2(\mathbb{R}) \) (contained in the \( \Gamma_\phi \) theta group). Then using the corollary to Theorem 5 of [3] we have

**Theorem 1.** Let \( \varphi \) be a \( \widetilde{\mathbb{K}} \times K \) finite function in \( \mathbb{F}_Q^+(s^2 - 2s) \) with \( s > \frac{1}{2}k \). Then the sum with \( (G, g) \in \overline{SL_2} \times O(\mathbb{Q}) \),

\[
T^L_\varphi(G, g) = \sum_{\xi \in L} \pi_Q(G, g)^{-1}(\varphi)(\xi),
\]

is absolutely convergent. Moreover, for \( (\Omega, \gamma) \in \Gamma^L(\mathbb{Q}) \times \Gamma_L(\mathbb{Q}) \), we have the functional equation

\[
T^L_\varphi(\Omega \gamma, g) = \sigma^L_\varphi(\Omega, \gamma) T^L_\varphi(G, g),
\]

where \( \sigma^L_\varphi \) is a unitary character on \( \Gamma^L(\mathbb{Q}) \times \Gamma_L(\mathbb{Q}) \) taking values in \( S_4 \) (where \( S_j = \{ z \in \mathbb{C} \mid z^j = 1 \} \) for \( j \) any positive integer). Moreover, \( T^L_\varphi \) is a \( C^\infty \) function on \( \overline{SL_2} \times O(\mathbb{Q}) \) satisfying \( D \ast T^L_\varphi(G, g) = T^L_\varphi(D^{-1}g, g) \) for any \( D \) in the

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universal enveloping algebra of \( \widetilde{\mathfrak{sl}}_2 \times O(Q) \) (\( * \) represents differentiation on the left). In particular, \( \omega_{\mathfrak{sl}_2} * T^L_\varphi = (s^2 - 2s) T^L_\varphi \). Finally we have the estimate

\[
|T^L_\varphi(G, g)| \leq M r_G^{s-1/2} ||g||^{-1} ||g+k/2-2
\]

where \( M \) is some positive constant independent of \((G, g)\), \( r_G \) denotes the A part of \( G \) in the Iwasawa decomposition of \( G = K_G a(r_G) n(x_G) \), and \( || \cdot ||_k \) denotes the Frobenius norm of a linear operator on \( \mathbb{R}^k \).

REMARK 1. The function \( T^L_\varphi \) is an automorphic form on \( \widetilde{\mathfrak{sl}}_2 \times O(Q) \) in the sense of the definitions in [1].

REMARK 2. The unitary character \( \sigma^L_\varphi \) on \( \Gamma^L(Q) \times \Gamma_L(Q) \) is given as 

\[
\sigma^L_\varphi(\Omega, \omega) = c(\Omega),
\]

where the map \( \Omega \mapsto c(\Omega) \) on \( \Gamma^L(Q) \) is given by 

\[
c\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, e = (\text{sgn} \, e) b_\delta \left( \frac{2\gamma}{\delta} \right)^k \left( \frac{D_Q(L)}{\delta} \right)
\]

where \( \gamma \neq 0 \) with

\[
b_\delta = \begin{cases} 
1 & \text{if } \delta \equiv 1 \mod 4, \\
\sqrt{-1} & \text{if } \delta \equiv 3 \mod 4,
\end{cases}
\]

and \((\cdot)\) the quadratic residue symbol as given in [4].

Using Remark 2 we then construct on \( P = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\} \), the upper half plane, a half-integral multiplier system for the discrete arithmetic group 

\[
\Delta_{N_L} = \left\{ \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \text{Sl}_2(\mathbb{Z}) \mid \gamma \equiv 0 \mod 2 N_L, \beta \equiv 0 \mod 2 \right\}
\]

doing \( \alpha \), taking values in \( S_4 \). That is: if \( \psi_Q(G) = (c(G, 1)))^{-1} \psi_2(G) \) with

\[
\psi_2(G) = \begin{cases} 
1 & \text{if } c_G \neq 0, \\
\text{sgn}(d_G) & \text{if } c_G = 0,
\end{cases}
\]

then

\[
v_Q(G_1 G_2)(c_3 z + d_3)^s = v_Q(G_1) v_Q(G_2)(c_1 z + d_1)^s (c_2 z + d_2)^s,
\]

where \( G_1, G_2 \in \Delta_{N_L} \) with

\[
G_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix} \quad \text{and} \quad G_1 G_2 = \begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix}
\]

(where \( z^s = |z|^s e^{\sqrt{-1} \text{arg} z} s \) with \(- \pi < \text{arg} z \leq \pi\)).

Then using Theorem 1 and the corollary to Theorem 5 of [3], we deduce the following.

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**Theorem 2.** Let \( \varphi \) be a function belonging to \( E_Q(s^2 - 2s, s, s_1, 0) \) (with \( s > \frac{1}{2}k \) and \( s_1 = s - \frac{1}{2}(a - b) \)) of the form on \( \Omega_+ : \varphi(X) = Q(X, X)^{s-1}e^{-\pi Q(X, X)}\|X_+\|^{-(s+s_1+k/2-2)}Q(X, \xi_+)^{s_1}, \)

where \( \xi_+ \in \mathbb{C}^d \) is a nonzero complex isotropic vector, i.e. \( Q(\xi_+, \xi_+) = 0 \). Then we let

\[
T^L_\varphi(z, g) = (\operatorname{Im} z)^{s/2} T^L_\varphi\left(\begin{bmatrix} x & y \\ 0 & x^{-1} \end{bmatrix}, 1\right), g
\]

(1.4)

with \( z = -y/x + \sqrt{-1}x^2 \in P \). Then we have the expansion

\[
T^L_\varphi(z, g) = \sum_{n \in \mathbb{Z}, n \geq 1} n^{s-1}e^{\pi \sqrt{-1} zn}\varphi_n^{s_1}(g),
\]

where

\[
\varphi_n^{s_1}(g) = \sum_{\{M \in L|Q(M, M) = n\}} Q(M, g)^{-1} \xi_+^{s_1} \|gM\_+\|^{-(s+s_1+k/2-2)}
\]

Then \( T^L_\varphi(z, g) \) is an antiholomorphic cusp form in \( z \) for \( \Delta_{NL} \) of degree \( |s| \) form for \( \Delta_{NL} \) with multiplier \( \nu_Q \) of degree \( s \), that is

\[
T^L_\varphi\left(\frac{az + b}{cz + d}, g\right) = \nu_Q(G)(cz + d)^{s}\bigg|_{L_\varphi}(z, g)
\]

with \( G = [a \ b \ c \ d] \in \Delta_{NL} \). Moreover, we have that \( \bigg|_{L_\varphi}(z, g) \) is a cusp form in \( z \) for \( \Delta_{NL} \), that is, \( \bigg|_{L_\varphi} \) is holomorphic at \( \infty \) (from 1.5) and \( T^L_\varphi (u + \sqrt{-1}v, g) = O(v^{-(s-1)/4}) \) as \( v \to 0 \) uniformly in \( u \).

**Remark 3.** Choosing the quadratic form \( Q = xy + zw \) on \( \mathbb{R}^4 \) and a suitable \( Q \) integral lattice \( L \subseteq \mathbb{R}^4 \), we obtain from the construction above automorphic forms similar to \( \varphi_{s_1}(\tau_1, \tau_2, z) \) in [6]. We also note the construction of related automorphic forms in [2] and [5] for the case \( k = 3 \).

**Remark 4.** From the invariance of \( \bigg|_{L_\varphi} \) in the \( O(Q) \) variable relative to \( \Gamma_L(Q) \), we see that \( \varphi_n^{s_1}(g\gamma) = \varphi_n^{s_1}(g) \) for all \( g \in O(Q), \gamma \in \Gamma_L(Q) \). The interpretation of formula (1.5) for \( \bigg|_{L_\varphi} \) is simply the Fourier expansion of \( \bigg|_{L_\varphi} \) at \( \infty \) with each Fourier coefficient \( \varphi_n^{s_1}(g)n^{s-1} \) an automorphic form for \( O(Q) \) relative to \( \Gamma_L(Q) \).

**Remark 5.** In a manner similar to the construction above (with the added assumption that \( b = 2 \)), we start with the function \( \varphi \in E_Q(s^2 + 2s, s, 0, s_2) \subseteq F_Q(s^2 + 2s) \) given by

\[
\varphi(Y) = |Q(Y, Y)|^{s-1}e^{\pi Q(Y, Y)}Q(X, \xi_-)^{-s_2} \quad \text{on } \Omega_-
\]

where \( s < -\frac{1}{2}k \) and \( s_2 = |s| + \frac{1}{2}(a - 1) \) and \( \xi_- \in \mathbb{C}^b \), nonzero complex isotropic, i.e. \( Q(\xi_-, \xi_-) = 0 \). Then as in Theorem 2 we let
\( (1.6) \quad \tilde{T}_{\psi}(z, g) = \sum_{n \in \mathbb{Z}, n \leq -1} |n|^{s-1} e^{\pi \sqrt{-1} n z} \tilde{\varphi}_{n}^{s}(g), \)

where

\[ \tilde{\varphi}_{n}^{s}(g) = \sum_{\{M \leq L \mid Q(M, M) = n\}} Q(M, g^{-1} \xi_{-})^{-s}, \]

\( z \in \tilde{\mathbb{P}} = \text{lower half plane}. \)

Then \( T_{\psi}^{L}(z, g) \) is an antiholomorphic cusp form in \( z \) for \( \Delta_{N_{L}} \) of degree \( |s| \) with multiplier \( v_{Q}. \)

Then we can analyze the cuspidal behavior of each \( \tilde{\varphi}_{n}^{s} \) determined in Remark 5.

**Theorem 3.** Let \( b = 2 \) and let \( \tilde{\varphi}_{n}^{s} \) be as in Remark 5. Then for the unipotent radical \( H \) of any rational maximal parabolic subgroup of \( O(Q) \) we have

\[ \int_{H/H \cap \Gamma_{L}(Q)} \tilde{\varphi}_{n}^{s}(gh) \, dh \equiv 0 \]

(with \( dh \) an \( H \) invariant measure on \( H/H \cap \Gamma_{L}(Q) \)) for all \( g \in O(Q) \) and all \( n \leq -1 \).

**Remark 6.** Theorem 3 implies that the family of automorphic forms \( \tilde{\varphi}_{n}^{s} \) belongs to the space of cusp forms (in the sense of \([1]\)) of \( L^{2}(O(Q)/\Gamma_{L}(Q)) \).

The case \( b = 2 \) turns out to be critical in the proof of Theorem 3. The basic idea behind the proof of Theorem 3 is what we call the Cusp Vanishing Theorem.

**Theorem 4.** Let \( \varphi \in F_{Q}(s^{2} + 2s) \) be a \( \tilde{K} \times K \) finite function with \( b = 2 \) and \( s < -\frac{1}{2} k \). Then for any \( X \in \Omega_{-} \) and for the unipotent radical \( H \) of any rational maximal parabolic subgroup of \( O(Q) \), \( \int_{H/H \cap \Gamma_{L}(Q)} \varphi(gh(X)) \, d\mu_{\chi}(h) \equiv 0 \) for all \( g \in O(Q) \) (with \( d\mu_{\chi} \) some \( H \) invariant measure on \( H/HX^{\chi}, HX^{\chi} = \text{isotropy group of } X \)).

Again we note the importance of the case \( b = 2 \). If \( b = 2 \), then \( O(Q)/K \) is a Hermitian symmetric space. We let \( F = f + p \) be the Cartan decomposition of the Lie algebra of \( O(Q) \). Then we have the direct sum \( \tilde{\mathfrak{c}}_{C} = \mathfrak{f}_{C} \oplus p^{+} \oplus p^{-} \), where \( p^{-} \) and \( p^{+} \) span the holomorphic and antiholomorphic tangent vectors at the “origin” in \( O(Q)/K \). Then we recall the construction of a family of holomorphic discrete series representations of \( O(Q) \). We consider \( K = O(a) \times O(2) \), and let \( \chi_{n}: K \to S^{1} \) be the unitary character on \( K \) which is trivial on \( O(a) \) and maps

\[ O(2) = \left\{ \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \, | \, -\pi < \theta \leq \pi \right\} \]
to $e^{\sqrt{-1}n\theta}$ $(n \in \mathbb{Z})$. Then we form the “holomorphic” unitarily induced representation space $H(O(Q)/K, \chi_n) = \{ \varphi: O(Q) \to \mathbb{C} | \varphi(gk) = \varphi(g)\chi_n(k) \}$ for all $g \in O(Q), k \in K, \varphi \ast W \equiv 0$ for all $W \in p^+$, and $\int_{O(Q)/K} |\varphi(g)|^2 \, d\sigma(g) < \infty$ with $\ast W$, convolution on the left and $d\sigma$ some $O(Q)$ invariant measure on $O(Q)/K$. Then we have

**Theorem 5.** The representation on $O(Q)$ in $A_s^-$ (see Remark 1 in [3]) is equivalent to the “holomorphic” induced representation of $O(Q)$ in $H(O(Q)/K, \chi_{s_2})$ where $s_2 = |s| + \frac{1}{2}a - 1$.

**Remark 7.** The representation of $O(Q)$ in $A_s^-$ (for $b = 2$) is thus always “square integrable”. Moreover $A_s^-$ is “integrable” if $s < 2 - k$.

**Corollary to Theorem 5.** Let $s < 2 - k$. Then each $\tilde{\varphi}_n^{s_2}$ given in Remark 5 is a “Poincaré series” on $O(Q)/\Gamma_L(Q)$. That is, there exists a $K$ finite function $q_n \in H(O(Q)/K, \chi_{s_2})$ so that $\tilde{\varphi}_n(g) = \sum_{\gamma \in \Gamma_L(Q)} q_n(\gamma g^{-1})$.

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