REAL ALGEBRAIC VARIETY STRUCTURES ON P. L. MANIFOLDS

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A closed smooth manifold $M^m$ is said to bound a smooth spine-manifold if $M$ bounds a compact smooth manifold $W^{m+1}$ and if there are a finite number of transversally intersecting closed submanifolds $\{M_i\}$ of $W$ such that $W/\bigcup M_i \approx \text{cone}(M)$, where $\approx$ is piecewise differentiable homeomorphism.

DEFINITION. An $A_1$-structure on a P. L. manifold $M^m$ is:

$M^m = M^m_0 \cup \bigcup_i \text{cone}(\Sigma_i) \times N_i$ where $M^m_0$ is a codimension zero smooth submanifold of $M$, $\partial M^m_0 = \bigcap_i \Sigma_i \times N_i$, $N_i$'s are smooth manifolds and $\Sigma_i$'s are exotic spheres bounding smooth spine-manifolds.

$A_1$-structures satisfy regular neighborhood and product structure properties, and there is a classifying space $B_{A_1}$ with inclusions $B_0 \rightarrow B_{A_1} \rightarrow B_{\text{PL}}$ (see [3]). This reduces the existence of $A_1$-structure on a P. L. manifold to a bundle lifting problem.

THEOREM 1. Any closed $A_1$-manifold is P. L. homeomorphic to a real algebraic variety.

COROLLARY 1. All P. L. manifolds of dimension less than 10 are P. L. homeomorphic to real algebraic varieties (also see [1]).

THEOREM 2. If a closed smooth manifold bounds a smooth spine-manifold, then it can be represented as a link of an isolated real algebraic singularity. (Converse of this is the Hironaka's resolution theorem.)

COROLLARY 2. Elements of $\Gamma_8$, $2\Gamma_{10}$, and all exotic spheres which admit fixed point free smooth involutions are links of real algebraic singularities (also see [2]).

A BRIEF SKETCH OF THE PROOFS. Let $M$ be a closed $A_1$-manifold. For simplicity assume $M^m = M^m_0 \cup \text{cone}(\Sigma^{m-1})$; then there is $W^m$ with closed submanifolds $\{M_i\}$ such that $W/\bigcup M_i \approx \text{cone}(M)$, and $\partial W = \Sigma$. Let $\tilde{M} = M^m_0 \cup W$.

By proving a relative version of the Nash-Tognoli approximation theorem we can make the smooth manifold $\tilde{M}$ a real algebraic variety $V$, so that the smooth submanifolds $\{M_i\}$ of $\tilde{M}$ correspond to the subvarieties $\{V_i\}$ of $V$.


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Let $V = f^{-1}(0)$ and $U_i = g^{-1}(0)$, where $f(x)$ and $g(x)$ are polynomials. Let $F(x, t) = f(x)^2 + (tg(x) - 1)^2$, then $\hat{F}(y) = |y|^2 d F(y/|y|^2) = 0$, $y = (x, t)$, $d = \text{degree } F$, gives the equations of

$$V/U_i \approx \tilde{M}/\cup M_i = M_0 \cup (W/\cup M_i) \approx M_0 \cup \text{cone}(\Sigma) = M.$$ 

This sketches the idea of the proofs of Theorem 1 and 2. Corollary 1 and 2 are true because elements of $\Gamma_8$, $2\Gamma_{10}$ bound spine manifolds (see [4]); and any exotic sphere $\Sigma$ with fixed point free smooth involution $\tau$ bounds the obvious spine manifold $\Sigma \times I/(x, 0) \sim (\tau(x), 0)$.

REFERENCES