We announce a number of single variable approximation theorems. Our approach is to extend de Branges' basic theory of Hilbert spaces of entire functions [2] to a Banach space setting. The resulting structure is sufficiently rich to provide both new approximation results and a unifying structure for many earlier results on approximation by entire functions which are related to the Bernstein approximation problem, for example, Akutowitz [1], Koosis [3], Levinson and McKean [5], Mergelyan [6], Pitt [7] and Pollard [8].

Let \( C_c \) be the space of continuous complex functions \( m(\lambda) \) on \( R^1 \) with compact support and the supremum norm \( |m| \). \( B \) denotes a fixed Banach function space on \( R^1 \) with (semi) norm \( \|f\| \). We assume that

1. \( C_c \cap B \) is dense in \( B \), and
2. The multiplication operator \( (m, f) \rightarrow m(\lambda)f(\lambda) \) is jointly continuous from \( C_c \times B \) into \( B \).

Examples of spaces satisfying (1) and (2) are \( L^p \) spaces, Orlicz spaces, Lorentz spaces \( L_{(p, q)} \) and spaces of continuous functions with weighted supremum norms. Because of condition (2) it follows that for \( f \in B \) and \( e \in B^* \), the linear functional on \( C_c \) given by \( m \rightarrow \langle mf, e \rangle \) is expressible in the form \( \langle mf, e \rangle = \int m(\lambda)d\mu_{f,e} \) where \( \mu_{f,e} \) is a unique finite Radon measure. The discrete spectrum \( \sigma_d(B) \) of \( B \) is the set \( \{\lambda: |\mu_{f,e}(\lambda)| > 0 \text{ for some } f \in B \text{ and } e \in B^* \} \).

Contained in \( B \) we fix a linear space \( H \) of entire functions with closure \( \overline{H} \). We assume for \( \text{Im } z \neq 0 \) and for \( f \) and \( g \) in \( H \) that the function

\[ F(\lambda) \equiv (z - \lambda)^{-1} \{f(z)g(\lambda) - g(z)f(\lambda)\} \in H. \]

If \( H \) is closed under the conjugation \( h \rightarrow \overline{h}(z) \) we call \( H \) symmetric. Two basic examples of symmetric \( H \) are the space \( P \) of all polynomials and the space \( F(T) \)

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of all Fourier transforms \( f(z) = \int f(\exp\{itz\})g(t)\,dt \) where \( g \) is infinitely differentiable and supported on \([-T, T]\). To avoid trivialities we assume

(4) for each \( \lambda \in \sigma_d(B) \) there exists an \( h \in H \) with \( h(\lambda) \neq 0 \).

The statement of our results require the auxiliary norm \( ||f||_+ = ||(\lambda - i)^{-1}f(\lambda)|| \) and the evaluation functionals \( \{e_z; z \in \mathbb{C}^1\} \) on \( H \) where \( e_z(f) = f(z) \) together with the norms \( L(z) = ||e_z|| \) and \( L^+(z) = ||e_z||_+ \).

**Theorem 1.** If \( L^+(\beta) = +\infty \) for some \( \beta \in \mathbb{R}^2^+ \) = \( \{z: \text{Im} z > 0\} \) then

\[(z - \lambda)^{-1}\overline{H} \subseteq \overline{H} \quad \text{for each } z \in \mathbb{R}^2^+.\]

**Theorem 2.** Let \( H_\beta = \{h \in H: h(\beta) = 0\} \). If \( H \) is symmetric and if \( (\beta - \lambda)^{-1}H_\beta \) is dense in \( H \) for some \( \beta \) with \( \text{Im}(\beta) \neq 0 \) then \( \overline{H} = B \) iff \( L^+(\beta) = +\infty \).

**Theorem 3.** If \( \beta \in \mathbb{R}^2^+ \) and \( 0 < L(\beta) < \infty \) then \( L(z) \) is continuous and subharmonic on \( \mathbb{R}^2^+ \). If in addition \( 0 < L(\gamma) < \infty \) for some \( \gamma \in \mathbb{R}^2^- \) then \( L(z) \) is continuous and subharmonic on \( \mathbb{C}^1 \).

Under the conditions of Theorem 3, \( \overline{H} \) is a closed subspace of entire functions \( f(z) \) satisfying \( |f(z)| \leq L(z)||f|| \) and (3).

**Theorem 4.** Assume \( H \) is closed and that \( L(z) \) is finite. Let \( K = B \cap \{f: f(z) \text{ is entire and } f(z)L^{-1}(z) \text{ is bounded on } \mathbb{C}^1\} \). Then \( H \subseteq K \) and the co-dimension of \( H \) in \( K \) satisfies \( \dim(K|H) \leq 1 \).

The case \( K = H \) is generic but \( \dim(K|H) = 1 \) can occur.

**Theorem 5.** Under the conditions of Theorem 4 there exist functions \( h_+ \) and \( h_- \) in \( H \) for which \( K \) consists of all entire functions \( f \in B \) satisfying

(i) \( f(z)h_+^{-1}(z) \) (resp. \( f(z)h_-^{-1}(z) \)) is analytic and of bounded type on \( \mathbb{R}^2^+ \) (resp. \( \mathbb{R}^2^- \)).

(ii) \( \sup f(\text{iy})L^{-1}(\text{iy}) < \infty, y \in \mathbb{R}^1 \).

These theorems can be refined when \( H = \mathcal{P} \) or \( H = F(T) \). The solutions of the Bernstein problem given in [1], [6], [8] are generalized to the present setting by

**Theorem 6.** If \( H = \mathcal{P} \) or \( H = F(T) \) then \( \overline{H} = B \) iff either of the equivalent conditions

(i) \( L^+(i) = +\infty \),

(ii) \( \int \log L^+(\lambda)(1 + \lambda^2)^{-1}d\lambda = +\infty \),

is satisfied.

The generalizations of the Paley-Wiener theorem given in [1], [4], [5] also hold in the present case. We set \( \overline{F}(T^+) = \bigcap\{\overline{F}(S): S > T\} \).

**Theorem 7.** Either \( \overline{F}(T^+) = B \) or \( \overline{F}(T^+) = B \cap E(T) \), where \( E(T) \) is
the space of entire functions of exponential type not greater than $T$.

When $B$ is a classical sequence space it may happen that both $L(z) < \infty$ and $H = B$. Series expansions for $L(z)$ are possible in this case and results related to classical interpolatory function theory may be obtained (see [7, p. 115]).

REFERENCES


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