Let $G$ be a reductive group defined over $\mathbb{Q}$. Index the parabolic subgroups defined over $\mathbb{Q}$, which are standard with respect to a minimal $^{(0)}P$, by a partially ordered set $\mathcal{P}$. Let 0 and 1 denote the least and greatest elements of $\mathcal{P}$ respectively, so that $^{(1)}P$ is $G$ itself. Given $u \in \mathcal{P}$, we let $^{(u)}N$ be the unipotent radical of $^{(u)}P$, $^{(u)}M$ a fixed Levi component, and $^{(u)}A$ the split component of the center of $^{(u)}M$. Following [1, p. 328], we define a map $^{(u)}H$ from $^{(u)}M(\mathbb{A})$ to $^{(u)}A = \text{Hom}(X^{(u)}M, \mathbb{R})$ by

$$\phi^{(u)}H(m) = |\chi(m)|, \quad \chi \in X^{(u)}M, m \in ^{(u)}(M).$$

If $K$ is a maximal compact subgroup of $G(\mathbb{A})$, defined as in [1, p. 328], we extend the definition of $^{(u)}H$ to $G(\mathbb{A})$ by setting

$$^{(u)}H(nmk) = ^{(u)}H(m), \quad n \in ^{(u)}N(\mathbb{A}), m \in ^{(u)}M(\mathbb{A}), k \in K.$$

Identify $^{(0)}a$ with its dual space via a fixed positive definite form $\langle \cdot, \cdot \rangle$ on $^{(0)}a$ which is invariant under the restricted Weyl group $\Omega$. This embeds any $^{(u)}a$ into $^{(0)}a$ and allows us to regard $^{(u)}\Phi$, the simple roots of $^{(u)}P$, $^{(u)}A$, as vectors in $^{(0)}a$. If $v \leq u$, $^{(v)}P \cap ^{(u)}M$ is a parabolic subgroup of $^{(u)}M$, which we denote by $^{(v)}P$ and we use this notation for all the various objects associated with $^{(v)}P$. For example, $^{(v)}a$ is the orthogonal complement of $^{(u)}a$ in $^{(v)}a$ and $^{(v)}\Phi$ is the set of elements $\alpha \in ^{(v)}\Phi$ which vanish on $^{(u)}a$.

Let $R$ be the regular representation of $G(\mathbb{A})$ on $L^2(ZG(\mathbb{Q})G(\mathbb{A}))$, where we write $Z$ for $^{(1)}A(\mathbb{R})^0$, the identity component of $^{(1)}A(\mathbb{R})$. Let $f$ be a fixed $K$-conjugation invariant function in $C_c^\infty(\mathbb{A} \backslash G(\mathbb{A}))$. Then $R(f)$ is an integral operator whose kernel is

$$K(x, y) = \sum_{\gamma \in G(\mathbb{Q})} f(x^{-1} \gamma y).$$

If $u < 1$ and $\lambda \in ^{(u)}a \otimes \mathbb{C}$, let $\rho(\lambda)$ be the representation of $G(\mathbb{A})$ obtained by inducing the representation

$$(n, a, m) \rightarrow ^{(u)}R_{\text{disc}}(m) \cdot e^{\langle \lambda, ^{(u)}H(m) \rangle}$$

from $^{(u)}P(\mathbb{A})$ to $G(\mathbb{A})$. Here $^{(u)}R_{\text{disc}}$ is the subrepresentation of the representation
of \((u)M(A)\) on \(L^2((u)A(\mathbb{R})^0 \cdot ((u)M(Q) \backslash (u)M(A))\) which decomposes discretely. We can arrange that \(\rho(\lambda)\) acts on a fixed Hilbert space \(((u)H\) of functions on 
\((u)N(A) \cdot ((u)A(\mathbb{R})^0 \cdot ((u)M(Q) \backslash G(A))\). If \(u = 1\), we take \((1)H\) to be the orthogonal complement of the cusp forms in the subspace of \(L^2(ZG(Q) \backslash G(A))\) which decomposes discretely.

**Theorem 1.** There exist orthonormal bases \((u)B\) of \((u)H, u \in \mathbb{Z}\), such that

\[K_E(x, y) = \sum_{u \in \mathbb{Z}} \int_{\left(\begin{array}{c} 1 \\ u \end{array}\right) \mathbb{A}} \sum_{\phi, \phi' \in (u)B} (\rho(\lambda, f)\phi', \phi)E(\phi, \lambda, x)\overline{E(\phi', \lambda, y)}d|\lambda|\]

converges uniformly for \(x\) and \(y\) in compact subsets of \(ZG(Q) \backslash G(A)\). (Here \(E(\phi, \cdot, \cdot)\) is the Eisenstein series associated with \(\phi\) as in [3, Appendix II].) Moreover, \(R_{\text{cusp}}(f)\), the restriction of the operator \(R(f)\) to the space of cusp forms, is of trace class, and if the Haar measures \(d|\lambda|\) on \((\begin{array}{c} 1 \\ u \end{array})\mathbb{A}\) are suitably normalized,

\[
\text{tr } R_{\text{cusp}}(f) = \int_{ZG(Q) \backslash G(A)} \left(K(x, x) - K_E(x, x)\right) dx.
\]

For any \(u \in \mathbb{Z}\), let \((u)\Phi\) be the basis of \((\begin{array}{c} 1 \\ u \end{array})\mathbb{A}\) which is dual to \((u)\Phi\). We write \(|u|\) for the number of elements in \((u)\Phi\) or \((u)\Phi\). Let \((u)\chi\) be the characteristic function of \(\{H \in (u)\mathbb{A} : \langle \mu, H \rangle > 0, \mu \in (u)\Phi\}\). Fix a point \(T \in (0)\mathbb{A}\) such that \(\langle \alpha, T \rangle\) is suitably large for each \(\alpha \in (0)\Phi\). Motivated by the results of [2, §9], we define

\[(\Lambda \phi)(x) = \sum_{u \in \mathbb{Z}} (-1)^{|u|} \sum_{\delta \in (u)P(Q) \backslash G(Q)} \int_{(u)N(Q) \backslash (u)N(A)} \phi(n\delta x)\ dn\]

for any continuous function \(\phi\) on \(ZG(Q) \backslash G(A)\). Let \(\widetilde{k}_T(x)\) and \(\widetilde{k}_E(x)\) be the functions obtained by applying \(\Lambda\) to each variable in \(K(x, y)\) and \(K_E(x, y)\) separately, and then setting \(x = y\). If \(\phi\) is a cusp form, \(\Lambda \phi = \phi\). From this it follows that

\[\widetilde{k}_T(x) - \widetilde{k}_E(x) = K(x, x) - K_E(x, x).
\]

**Theorem 2.** The functions \(\widetilde{k}_T(x)\) and \(\widetilde{k}_E(x)\) are both integrable over \(ZG(Q) \backslash G(A)\), and the integral of \(\widetilde{k}_T(x)\) equals

\[
\sum_{u \in \mathbb{Z}} \int_{\left(\begin{array}{c} 1 \\ u \end{array}\right) \mathbb{A}} \sum_{\phi, \phi' \in (u)B} (\rho(\lambda, f)\phi', \phi) \int_{ZG(Q) \backslash G(A)} \Lambda E(\phi, \lambda, x) \cdot \overline{\Lambda E(\phi', \lambda, x)} d|\lambda|.
\]

It should eventually be possible to calculate the integrals in Theorem 2 by extending the methods of [2, §9]. On the other hand, \(\widetilde{k}_T(x)\) is not a natural truncation of \(K(x, x)\). This defect is remedied by the following
THEOREM 3. The function
\[ k^T(x) = \sum_{u \in \mathfrak{t}} (-1)^{|u|} \sum_{\delta \in \{(u)_{P(G) \backslash G(Q)}^{(N(A) \mu)}} f(x^{-1} \delta^{-1} \mu u \delta x) dn 
\]
\[ \cdot \langle(u)_{\mathfrak{X}}(u)H(\delta x) - T \rangle \]
is integrable over $ZG(G) \backslash G(A)$. For sufficiently large $T$, the integrals over $ZG(Q) \backslash G(A)$ of $k^T(x)$ and $\tilde{k}^T(x)$ are equal. □

The proofs will appear elsewhere.

REFERENCES

