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BOOK REVIEWS

Modern algebra with applications, by William J. Gilbert, Wiley, New York, 1976, xii + 348 pp., $21.95.

The book under review can only be described as an elementary text book in algebra. Thus it may seem strange that it is being reviewed in these pages which are usually reserved for the discussion of scholarly books or some advanced graduate texts. In fact when I was asked to write this review my own reaction was one of surprise.

However, there are some cogent reasons why it, or some book like it, should be discussed here. It is clear to me that the publication of these books, and the fact that they are being written, reflect a change that is taking place in emphasis and direction in the teaching of algebra to our beginning students. Perhaps this is even symptomatic of a trend in the kind of research that is being done, or will be done, in algebra, namely, a greater concern for the use of recent algebraic results and a lesser concern for the discovery of such new results. In the past few years there have been some prominent voices saying that the boom days of abstract algebra are over for now and that the algebra that will be done in the near future will be rather more concrete, with a sharper emphasis on the solution of specific problems. The appearance of these elementary books with "application" or some isomorph in their titles can be viewed as one more bit of evidence that there is some undercurrent in the algebraic community that things are not as they used to be.

Be that as it may, one can legitimately ask: why discuss a book of such an elementary nature here? If the trend in teaching will be to slant the material towards applications, appropriate texts will be needed. It is safe to say that the definitive books of this nature have yet to be written. The natural candidates to write these definitive works are practicing algebraists, people who would normally read the reviews here. By some discussion of what has already been written, or what might be appropriate to write, maybe some algebraists could become concerned enough to try their hands at writing these texts. With a little luck and lots of failed attempts we might eventually have some really good books come out.

Generally, the few books of this applied character that have been published so far fall into one of relatively few types. The first type–and this would most aptly describe Gilbert's book–tries to develop the algebra as if one were teaching the usual first course in algebra, however taking excursions outside to make applications of the algebra developed.

The second type, of which the book by Birkhoff and Bartee is the best known example, sets for itself as prime goal the applications rather than the development of the algebra *per se*. The intended readership is not so much the mathematics students but, rather, users of algebra–computer scientists, physicists, electrical engineers, economists, and others.

The basic difference between these two types of books is one of emphasis and general philosophy rather than content. In the final analysis, however, except for the depth to which the applications are explored, their similarities tend to outweigh their differences.

A third type–and the one that seems to me to have been the most successful to date–concentrates on some particular slice of algebra, develops what it needs about the algebraic structure to be used, and uses these results in a variety of interesting applications. Two very good examples of this kind of book come to mind. One is the book by Ben Noble on linear algebra; he gives a rich assortment of varied nontrivial uses of linear algebra in a diversity of situations. A much more sophisticated book is that of Willard Miller on applications of symmetry groups. This lovely book–in my opinion the best of its kind–gives a spectrum of possible exploitations of group theory to physics. It is not intended for beginners, and quickly gets into difficult mathematics. Clearly it cannot be used as a beginning text in algebra. But it does exemplify what one would hope could be achieved at a lower level.

While the two books cited above are too narrow in their algebraic scope for a first text in abstract algebra and get into material far too hairy for most neophytes, they serve as a *source* for material to illustrate the applications of some of the algebra that is expounded in the usual introductory course in algebra. In fact this should be the fundamental purpose of this third category of books, at least in our teaching endeavors, that is, to serve as source material for the regular algebra course.

I would hope that other people would try to do for other parts of algebra what Noble and Miller have done for linear algebra and group theory (at a level somewhat more accessible to the very beginner). One could use the relevant parts of Birkhoff and Bartee to illustrate the use of finite fields in the area of error correcting codes. It would be healthy to have other books on these topics mentioned above, and some that would give applications of elementary commutative ring theory, say, to combinatorics and number theory, and of the other algebraic stuctures that arise in the regular algebra course.

Still another type of book with applications has appeared. I would describe this last category as rather cook-booky. It picks a certain theorem, or class of theorems, states it or them without proof and without giving any inkling as to why these results are valid, and proceeds to show how to use these results by laying out a sort of "do-it-yourself" program. While these books are intended to be used in conjunction with more "respectable" algebraic texts, I find their philosophy distasteful and their possible misuse dangerous. They can easily degenerate into "mathematics by rule of thumb". The material they consider is far too fragile and far too subtle to be used blindly in a programmed way without an understanding of what really is involved.

Let me get down to the business at hand, namely Gilbert's book. It is well written, but to my mind his attempt simultaneously to develop some algebra and to show how it is used just does not come off. Outside of some of the material on field theory the algebra goes too slowly and seems to get nowhere. In what might be described as the theoretical part there is too much superficiality, so much so that the book could not adequately serve as a text for the regular algebra course. In its applied parts it suffers from the same sort of superficiality. He doesn't cut deeply enough into these applications to enlighten this reader as to what all the shouting is about. If one compares the treatment here and in Birkhoff and Bartee of the topics they have in common, there just is no comparison; the latter book does it better, and goes further and deeper. Some topics are introduced, cursorily discussed and abandoned without obtaining even the semblance of a meaningful statement (never mind theorem) about the situation being expounded upon. This is perhaps most fragrantly illustrated by the discussion of crystallographic groups. They are defined, and that's it. Outside of some *pro forma* bow to applications what has been accomplished for the reader by these few words about crystallographic groups? At best, nothing; at worst a bafflement on the student's part: "so what?".

Perhaps this state of affairs is inevitable. In order to get some meaningful and incisive applications one needs an arsenal of meaningful and incisive results. Can one develop a subject to a nontrivial point, get some deep results, see how they are used effectively–all this in one shot, in a short period of time (one year), in a book of manageable size? At one time I would have said yes, that a very skillful organization of material and some very lucid writing could pull it off. Now I have serious doubts that this state of bliss can be achieved.

On the other hand, I feel that we have been remiss in the past in teaching our beginners lots of algebra without giving them the slightest indication of how this algebra can be used. The usual argument was that we didn't have enough time to teach all the abstract stuff and, in addition, applications. This is certainly true, we don't have enough time for all this. So, something has to give. Why not teach fewer abstract topics the first time around? What will be lost by this? One can always pick up the things left out both better and faster at the first-year graduate level when the students are more experienced, more sophisticated, and much more at home with abstract mathematics. In the time created by the omission of topics we could do applications of the algebra that has been taught. But let these applications be convincing and not contrived, otherwise we'll turn the students off and make them cynical both about the applications and the underlying algebra.

As I indicated earlier, I am somewhat pessimistic about the possibility of writing an honest introductory text in algebra which is heavily and meaningfully laced with applications. In my opinion the best solution to the problem of introducing good applications into the algebra instruction is to have a compendium of good books, each concentrating on how a different part of algebra can be successfully exploited in a host of contexts. On some topics such books already exist, but more and better ones would be welcome. For many parts of algebra none–not even bad ones–are as yet to be found. I would hope that in the next few years some algebraist will be inspired to write such books.

Let me close by grinding a few axes. I would love to see some elementary book on linear algebra or on applications of linear algebra in which proofs would be given of the beautiful theorems of Frobenius on matrices with nonnegative entries. These results lend themselves readily to applications in Markov chains, stochastic matrices, models in economics, and many other places. Special cases of these general theorems have been "discovered" and attributed to people working in these varied, allied fields, fifty or more years after they were so definitively obtained by Frobenius. The very insightful proof by Wielandt using a minimax argument gives a ready entry into the arena of these results and makes all these results accessible to reasonable juniors and seniors.

The second axe that I have to grind is that, with all the hullabaloo about applications of algebra to economics, genetics, physics, electrical engineering, and so on, little attempt is made, at the early level, to show how algebra can be applied in *mathematics* itself. Sure, using beginning field theory, or some Galois theory, the question of constructibility by straight edge and compass, or the insolvability of the quintic are presented. Outside of these, very little effort is made to illustrate how algebra, even elementary algebra, can be successfully employed in other parts of mathematics. With a modicum of commutative ring theory nice results in number theory can be obtained. Very few of our students see how configurations in a projective plane translate into algebraic statements about the ring of coordinates of the plane, and how, once this is done, theorems in geometry can be proved by proving theorems about these associated rings. There have been resounding successes in combinatorics using deep results in commutative ring theory. By concentrating on specialized situations of these one might be able to get nice applications of very elementary commutative ring theory to interesting (albeit special) problems in combinatorics. One of my own favorite applications of algebra to number theory is Schur's argument for the Gaussian sums, using the fact that the trace of a matrix A is the sum of its characteristic roots and that the characteristic roots of A^2 are the squares of those of A, together with a hand-dirtying argument (which is healthy for our students to see) at the end.

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Integral geometry and geometric probability, by Luis A. Santaló, Addison-Wesley Publishing Company, Reading, Mass., 1976, xvii + 404 pp., $ 19.50.

Integral geometry was the name coined by Wilhelm Blaschke in 1934 for the classical subject of geometric probability. During that year the author came to Hamburg from Madrid and the reviewer from China, and we sat in Blaschke's course on geometric probability. The main reference was an "Ausarbeitung" of a course by the same name given by G. Herglotz in Göttingen. At the end of 1934 the author found his now famous proofs of the isoperimetric inequality in the plane and Blaschke himself found the fundamental kinematic formula and started a series of papers under the general title of "integral geometry". It was a fruitful and enjoyable year for all concerned.

Integral geometry is exactly 200 years old if we identify its birth with Buffon's solution in 1777 of the needle problem: A needle of length h is placed at random on a plane on which are ruled parallel lines at a distance $D \geqslant h$ apart. Find the probability that it will intersect one of these lines. In fact, the answer is $p = 2h/\pi D$. Experiments were made to determine π on the basis of this result, usually with great accuracy.

Elementary problems on geometric probability are many and are interesting. But until 1928 J. L. Coolidge still held the opinion that the subject is