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NONCOMMUTATIVE ERGODIC THEOREMS

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In [4], [5] E. C. Lance extended the pointwise ergodic theorem to actions of the group of integers on von Neumann algebras. Our purpose is to extend other pointwise ergodic theorems to von Neumann algebra context: the Dunford-Schwartz-Zygmund pointwise ergodic theorem (Theorem 3), the pointwise ergodic theorem for amenable locally compact connected groups (Theorem 4), the Wiener local ergodic theorem for $\mathbf{R}_+^d$ (Theorem 5) and for general Lie groups (Theorem 6).

We fix a pair $(\mathfrak{A}, \rho)$ where $\mathfrak{A}$ is a von Neumann algebra and ρ is a faithful normal state on $\mathfrak{A}$, we call *kernel* a positive linear contraction T of $\mathfrak{A}$ into itself such that $\rho(1) = 1$, $\rho(Ta) = \rho(a)$, $\rho((Ta)^*Ta) \leq \rho(a^*a)$ $\forall a \in \mathfrak{A}$. The set K of kernels of $\mathfrak{A}$ is a semigroup.

Let G be a locally compact semigroup, we call measurable (resp. continuous) representation of G into $\mathfrak{A}$ every (semigroup) homomorphism τ of G into K such that the mapping $G \ni g \longrightarrow \tau_g a \in \mathfrak{A}$ is ultrastrongly measurable (resp. continuous) for $a \in \mathfrak{A}$. The system $(\mathfrak{A}, \rho, G, \tau)$ is called a measurable (resp. continuous) *noncommutative dynamical system* (see [1], [5]).

In what follows, we assume that G admits a left invariant Haar measure ν. We call *amenable sequence* of G an increasing sequence $(V_n)_{n \geq 1}$ of Borel subsets of G such that $\bigcup_{n\geq 1} V_n$ generates G and

$$\sup_{g \in C} \frac{\nu(V_n \,\Delta\, gV_n)}{\nu(V_n)} \xrightarrow[n\to\infty]{} 0,$$

for every compact subset K' of C (cf. [3]).

We note $\mathfrak{A}^\tau = \{a \in \mathfrak{A} | \tau_g a = a \ \forall g \in G\}$.

Following E. C. Lance (cf. [4], [5]), we say that a sequence a_n of $\mathfrak{A}$ converges *almost everywhere* (or *almost uniformly*) to an element a of $\mathfrak{A}$ if for every $\epsilon > 0$, there exists a projection $e \in \mathfrak{A}$ such that $\rho(e) \geq 1 - \epsilon$ and $\|(a_n - a)e\| \longrightarrow 0$, $n \longrightarrow \infty$ (when $\mathfrak{A}$ is commutative, $\mathfrak{A} = L^\infty(X, \nu)$, with (X, ν) a probability space, this convergence coincides with the *almost everywhere pointwise convergence* via Egorov's theorem).

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The following theorem generalizes that of E. C. Lance (cf. [5, Theorem 4.2.]).

THEOREM 1 (UNIFORM ERGODICITY). *Let* $(\mathfrak{A}, \rho, G, \tau)$ *be a measurable dynamical system,* $(V_n)_{n \geqslant 1}$ *an amenable sequence of* G. *The set*

$$\mathfrak{A}_u = \{a \in \mathfrak{A} \mid \left\| \frac{1}{\nu(V_n)} \int_{V_n} \tau_g a \, dg - \phi_\tau(a) \right\| \xrightarrow[n \to \infty]{} 0\}$$

is ultrastrongly dense in $\mathfrak{A}$, *where* ϕ_τ *is the canonical projection of* $\mathfrak{A}$ *onto* $\mathfrak{A}^\tau$.

The following maximal lemma generalizes that of E. C. Lance for the case of **Z** (cf. [5, Lemma 5.2]).

THEOREM 2 (MAXIMAL LEMMA). *Let* τ^i, $i = 1, \ldots, d$ *be* d *measurable representations of* $\mathbf{R}_+$ *into* $(\mathfrak{A}, \rho)$. *For every* $a \in \mathfrak{A}_+$, $\|a\| \leqslant 1$ *and* $\epsilon > 0$ *such that* $\rho(a) = \epsilon$, *there exist* $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $c \in \mathfrak{A}_*$, $\|c\| \leqslant \gamma$, $\rho(c) \leqslant \alpha\epsilon^\beta$ *satisfying*

$$\frac{1}{T_1 \times \cdots \times T_d} \int_0^{T_1} dt_1 \int_0^{T_2} dt_2 \cdots \int_0^{T_d} dt_d \, \tau^1_{t_1} \tau^2_{t_2} \cdots \tau^d_{t_d} a \leqslant c,$$

for all $T_i > 0$, $i = 1, \ldots, d$. *The constants* α, β *and* γ *depend only on the dimension* d.

Combining the above results and Lance's method in [5], we obtain, the following extensions of the Dunford-Schwartz-Zygmund theorem (cf. [2], [7]) and the Emerson-Greenleaf theorem (cf. [3]):

THEOREM 3 (ALMOST UNIFORM ERGODIC THEOREM FOR $\mathbf{R}^d_+$). Let τ^i, $i = 1, \ldots, d$ *be as in Theorem* 2. *For every* $a \in \mathfrak{A}$, we have

$$\frac{1}{T_1 \times \cdots \times T_d} \int_0^{T_1} dt_1 \cdots \int_0^{T_d} dt_d \, \tau^1_{t_1} \cdots \tau^d_{t_d} a \longrightarrow \phi_{\tau^1} \cdots \phi_{\tau^d}(a)$$

almost everywhere as $T_i \longrightarrow \infty$, $i = 1, \ldots, d$.

(We also have a discrete version of Theorems 2 and 3.)

THEOREM 4 (ERGODIC THEOREM FOR AMENABLE CONNECTED GROUPS). *Let* $(\mathfrak{A}, \rho, G, \tau)$ *be a dynamical system with* G *an amenable locally compact connected group. There exists an amenable sequence* $(V_n)_{n \geqslant 1}$ *of* G *such that for every* $a \in \mathfrak{A}$, *we have*

$$\frac{1}{\nu(V_n)} \int_{V_n} \tau_g a \, dg \xrightarrow[n \to \infty]{} \phi_\tau a \qquad \textit{almost everywhere.}$$

We also have two extensions of Wiener's local theorem (cf. [6]):

THEOREM 5 (LOCAL ERGODIC THEOREM FOR $\mathbf{R}_+^d$). *Let τ^i, $i = 1, \ldots, d$, be d continuous representations of $\mathbf{R}_+$ into $(\mathfrak{A}, \rho)$. For every $a \in \mathfrak{A}$, we have*

$$\frac{1}{T_1 \times \cdots \times T_d} \int_0^{T_1} dt_1 \cdots \int_0^{T_d} dt_d \, \tau_{t_1}^1 \cdots \tau_{t_d}^d a \longrightarrow a$$

almost everywhere as $T_i \longrightarrow 0$, $i = 1, \ldots, d$.

THEOREM 6 (LOCAL ERGODIC THEOREM FOR LIE GROUPS). *Let $(\mathfrak{A}, \rho, G, \tau)$ be a measurable dynamical system with G a Lie group. There exists a decreasing sequence $(W_n)_{n \geq 1}$ of neighborhoods of the identity 1_G, $W_n \searrow \{1_G\}$ such that for every $a \in \mathfrak{A}$, we have*

$$\frac{1}{\nu(W_n)} \int_{W_n} \tau_g a \, d\nu(g) \underset{n \to \infty}{\longrightarrow} a \quad \text{almost everywhere.}$$

Detailed proofs of these results will be published elsewhere.

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