DETERMINATION OF THE AUGMENTATION TERMINAL
FOR FINITE ABELIAN GROUPS

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Let $G$ be a finite abelian group, and let $IG$ denote the augmentation ideal in the integral group ring $ZG$. The graded ring associated with the filtration on $ZG$ determined by the powers of $IG$ is

$$\text{gr } ZG = \sum_{n \geq 0} \bigoplus IG^n/IG^{n+1}.$$ 

We write $Q_n G = IG^n/IG^{n+1}$. As is well known [1], [6], the sequence $Q_n G$ becomes stationary after a finite number of steps. We call its terminal value the augmentation terminal, $Q_\infty G$. We outline here a method for investigating $Q_\infty G$ for any $G$.

An obvious splitting allows us to assume that $G$ is a $p$-group.

We choose a generator for each cyclic direct factor of $G$. Let $\Gamma$ be our set of such generators, and let $\Lambda = \{\lambda | \lambda + 1 \in \Gamma \}$. Generalizing Lemma 2 of [3] we have

**Lemma.** For $n \geq 1$ the set of $n$-fold products of elements of $\Lambda$ generates $IG^n$; a fortiori it generates $Q_n G$.

If $\lambda \in \Lambda$ there is an integer $r$ such that $(\lambda + 1)^{p^r} - 1 = 0$. Furthermore, by the structure of $G$, these equations are the only possible source of relations among the elements of $\Lambda$. Hence we have immediately

**Theorem 1.** Let $f(\lambda_1, \ldots, \lambda_k)$ be a nontrivial relator in $Q_n G$, where the $\lambda_i \in \Lambda$. Let $\lambda_i + 1$ be of order $p^{r_i}$ in $G$, each $i$. Let $X_1, \ldots, X_k$ be indeterminates over $Z$. Then there are polynomials $h_i(X_1, \ldots, X_k)$ with integer coefficients such that

$$f(X_1, \ldots, X_k) - \sum_{i=1}^{k} ((X_i + 1)^{p^{r_i}} - 1)h_i(X_1, \ldots, X_k)$$

has no terms of degree $\leq n + 1$.

Actually using this result to find relators is far from easy, as the references show [2], [7], [8]. If $G$ is an elementary $p$-group we have

**Theorem 2.** The relators in $Q_\infty G$ are generated by $\{p^\lambda | \lambda \in \Lambda \}$ and $\{p^\mu \lambda - \lambda p^\mu | \lambda, \mu \in \Lambda \}$. 

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PROOF. Easy hand calculation shows that these are indeed relators. It is trivial to work out the group given by these relators and then use Theorem 5 of [6] (see also Theorem 3 below).

By much more tedious calculation one can show

LEMMA. If \( \lambda, \mu \in \lambda \) and \( \lambda + 1, \mu + 1 \) have order \( p^2 \) in \( G \), then \( \lambda p^2 \mu p^2 - \lambda p^{p^2} \mu^p \) is a relator.

The appropriate generalization is readily conjectured, but a direct proof is likely to be very difficult.

We developed in [3], [7] and [8] a technique of "standard forms" which is generally suitable for determining the structure of \( IG^n \) modulo a given set of relators of \( Q_n G \). This technique may be applied to any \( G \). If the order of the group so determined is the same as that of \( Q_\infty G \), then \( Q_\infty G \) has been found. Otherwise, another relator must be hunted down.

The order of \( Q_\infty G \) is calculated via the module index \( [B \cap Q : IG] \) where \( B \) is the maximal order in \( QG \). This follows from [6], where we used our results on invertible powers of ideals [4], [5]. Specifically, by direct calculation from Theorem 5 of [6].

THEOREM 3. Let \( G \) be the direct product of \( a_i \) cyclic groups of order \( p_i^t \), \( 1 \leq i \leq m \). Then the order of \( Q_\infty G \) is \( p^J \), where

\[
p^J = p^{t_1} + \cdots + p^{t_m} - 1
\]

in which

\[
t_i = a_1 + 2a_2 + \cdots + (i - 1)a_{i-1} + i(a_i + \cdots + a_m - 1), \quad 1 \leq i \leq m.
\]

BIBLIOGRAPHY

2. David Ford and Michael Singer, Relations in \( Q_n(Z_4 \times Z_3) \) and \( Q_n(Z_8 \times Z_3) \), Comm. Algebra 5 (1977), 83–86.