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EQUIVARIANT SMOOTHING THEORY

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Given a finite group G acting on a topological manifold M, when can we put a smooth structure on M such that G acts smoothly? Our approach to this problem is via equivariant immersion theory. This generalizes the immersion theory approach of [12], and we begin by reviewing these ideas. Details will appear in [13].

1. The immersion approach to smoothing theory. A map $\alpha: M_1^n \to M_2^n$ between n-dimensional topological manifolds is called a (topological) *immersion* if α is a local homeomorphism. Of course, a smooth immersion is a topological immersion of the underlying topological manifolds. The basis of the immersion approach to smoothing is the following trivial lemma:

LEMMA 1. *A topological immersion α of a topological manifold M^n into a smooth manifold V^n defines a unique smooth structure on M such that α becomes a smooth immersion.*

In fact, define smooth local coordinates on M by pulling back the local coordinates on V via the local homeomorphisms. We will denote this smooth structure by M_α.

Recall that the differential of a smooth immersion $f: V_1^n \to V_2^n$ induces a bundle homomorphism $df: TV_1 \to TV_2$ of the tangent vector bundles which is an isomorphism on fibres. Call such a bundle homomorphism a representation and let $R(TV_1, TV_2)$ be the space of representations with the C^0-topology and $I^\infty(V_1, V_2)$ the space of smooth immersions with the C^∞-topology. The Smale-Hirsch theorem for manifolds of the same dimension states:

THEOREM A (HIRSCH). *If no component of V_1 is closed, $d: I^\infty(V_1, V_2) \to R(TV_1, TV_2)$ is a weak homotopy equivalence. The relative version for immersions modulo a given immersion on a neighborhood of a closed subset A holds, provided $\overline{M - A}$ has no compact components.*

For a topological manifold M we have Milnor's tangent microbundle [15], [12]. Since the fibre of τM over $p \in M$ is essentially a neighborhood germ, a local homeomorphism $f: M_1 \to M_2$ defines a microbundle representation

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$df\colon \tau M_1 \to \tau M_2$. (Explicitly, the total space of τM is any neighborhood U of the diagonal in $M \times M$ and $df = f \times f | U$, U sufficiently small.) Lees' topological immersion theorem [14] for manifolds of the same dimension states:

THEOREM B. *If no component of M_1 is closed, $d\colon I^t(M_1, M_2) \to R^t(\tau M_1, \tau M_2)$ is a weak homotopy equivalence.*

Here the "space" $I^t(M_1, M_2)$ of topological immersions must be treated as a simplicial set and similarly for $R^t(\tau M_1, \tau M_2)$ [12]. Since each n-dimensional microbundle contains an essentially unique R^n bundle, and these two categories of bundles are equivalent by Kister's theorem [10], we can also consider $R(\tau M_1, \tau M_2)$ to be the singular complex of the space of R^n bundle representations. Lees' theorem is proved following the scheme of Haefliger and Poenaru [5] for piecewise linear immersions after proving a topological isotopy extension theorem based on the work of Kirby [8].

By taking essentially the smooth singular complex $I^s(V_1, V_2)$ of $I^\infty(V_1, V_2)$ and the singular complex $R^s(TV_1, TV_2)$ of $R(TV_1, TV_2)$ we get a homotopy commutative diagram:

$$\begin{array}{ccc} I^s(V_1, V_2) & \xrightarrow{\ d\ } & R^s(TV_1, TV_2) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \phi} \\ I^t(V_1, V_2) & \xrightarrow{\ d\ } & R^t(\tau V_1, \tau V_2) \end{array}$$

where F is obtained by forgetting the smooth structure and ϕ by embedding TV as a neighborhood of the diagonal in $V \times V$ via the exponential map and observing that the topological differential and smooth differential then agree up to a natural homotopy.

As an example, if τM^n is trivial, i.e., equivalent to $M \times R^n$, we can obviously construct a microbundle representation of τM into τR^n. By Theorem B, if M is open, there is a topological immersion $\alpha\colon M \to R^n$, which defines a smooth structure M_α on M by Lemma 1.

More generally (and avoiding technicalities), if τM contains a vector bundle ξ and U is a contractible open set of M, $\xi | U$ is trivial and we have a vector bundle representation $\xi | U \to TR^n$ and hence a microbundle representation $\tau U = \tau M | U \to \tau R^n$, which induces a smoothing of U. Further, because the smoothing of U corresponds to the trivialization of $\xi | U$, if U' is another such neighborhood, the smoothing of $U \cap U'$ can be extended to a smoothing of U' corresponding to $\xi | U'$. That is, by Theorem A (relative version), there is a smooth immersion f of $U \cap U'$ in R^n whose differential extends to a vector bundle representation of $\xi | U' \to TR^n$. By Theorem B (relative version), f extends to a topological immersion $f'\colon U' \to R^n$ which induces a smooth structure on U' extending that on $U \cap U'$. Thus by induction over a countable open cover we get a smoothing of M corresponding to the reduction ξ of τM, provided M is open.

Define two smooth structures M_α, M_β on a topological manifold M to be *isotopic* if id_M is ambient isotopic as a homeomorphism of M_α onto M_β to a diffeomorphism. Then in [12] (see also [9]), we prove for general (in particular, closed) M:

THEOREM C. *If $n \neq 4$, the isotopy classes of smoothings of M^n are in bijective correspondence with the homotopy classes of reductions of τM to a vector bundle.*

The condition $n \neq 4$ comes from the fact that the immersion theorem does not apply to closed manifolds so that we have to apply it to $M - p$. In order to extend the smoothing over p, and to prove uniqueness up to isotopy, the smoothing near p has to be "straightened out" and this requires engulfing techniques which hold for $n \geqslant 5$. The case $n \leqslant 3$ is classical.

Now homotopy classes of reductions of τM correspond to homotopy classes of lifts of the classifying map $\tau\colon M \to B\ \mathrm{Top}_n$ of the tangent R^n bundle to BO_n. Here Top_n is the group of homeomorphisms of R^n with the C^0-topology and O_n is the orthogonal group. The map of classifying spaces $BO_n \to B\ \mathrm{Top}_n$ may be considered as a fibre space with fibre Top_n/O_n. Thus the obstructions to smoothing and uniqueness lie in $\pi_i(\mathrm{Top}_n/O_n)$, $i \leqslant n$.

The analogue of the fact that $O_{n+1}/O_n = S^n$ is the result [11] that $\mathrm{Top}_{n+1}/\mathrm{Top}_n = S^n \times BC(S^n)$. The group $C(S^n)$ is the pseudoisotopy or concordance group of S^n; i.e., homeomorphisms of $I \times S^n$, $I = [0, 1]$, which are the identity on $0 \times S^n$. Thus we have a homotopy theoretic fibration $\mathrm{Top}_n/O_n \to \mathrm{Top}_{n+1}/O_{n+1}$ with fibre $C(S^n)$. For $n \leqslant 3$ every manifold has a unique smoothing up to isotopy. For $n \geqslant 5$, it can be shown that $\pi_i C(S^n) = 0$ for $i \leqslant n + 1$. In fact, by surgery arguments of [7] and [16], $\pi_i C(S^n) = \pi_i C^{pl}(S^n)$, the piecewise linear group. The result then follows from Haefliger and Wall's analysis of $\pi_i PL_{n+1}/PL_n$, see [6]. Hence

$$\pi_i(\mathrm{Top}_n/O_n) = \pi_i(\mathrm{Top}/O), \qquad i \leqslant n+1,$$

where

$$\mathrm{Top} = \operatorname{ind}\lim_{n\to\infty} \mathrm{Top}_n \text{ and } 0 = \mathrm{Lim}\ O_n$$

under inclusion. Finally, the computation of $\pi_i\mathrm{Top}/O$ can be reduced to computing homotopy groups of spheres by surgery methods. In principle, therefore, one can compute the obstruction groups.

2. Equivariant smoothing. Let G be a finite group. A topological or smooth G-immersion of G-manifolds is just an immersion which is a G-map. The equivariant version of Lemma 1 is:

LEMMA 1 EQ. *A topological G-immersion α of a topological G-manifold M^n into a smooth G-manifold V^n defines a unique equivariant smooth structure M_α on M such that α becomes an equivariant smooth immersion.*

If V is a smooth G-manifold, the differential of the action of G on V induces an action of G on TV making it into a G-vector bundle [3] and [17]:

DEFINITION. A *G-vector bundle* is a vector bundle $p\colon E \to B$ where E and B are G-spaces, p is a G-map, and the action of G on E is through vector bundle maps.

The differential of a smooth G-immersion $f\colon V_1^n \to V_2^n$ induces a G-bundle homomorphism $df\colon TV_1 \to TV_2$ which is an isomorphism of fibres. Let $R_G(TV_1, TV_2)$ be the space of G-vector bundle representations and $I_G^\infty(V_1, V_2)$ the space of G-immersions. Bierstone [3] has given an equivariant Gromov theory proving in particular a G-version of Theorem A. To state it we first need the definitions:

DEFINITION (BREDON [4]). A topological G-manifold M is called *locally smooth* if M has an atlas of G-invariant open sets U, such that each U admits an equivariant smoothing.

DEFINITION. Let $M_{(H)}$ be the union of orbits of type (H). $M_{(H)}$ is G-invariant and a bundle over $M_{(H)}/G$ with fibre G/H [4]. If M is a (locally) smooth G-manifold, $M_{(H)}$ is a (locally) smooth submanifold. We say M satisfies the *Bierstone Condition* if no G-component of $M_{(H)}$ is a closed manifold. (A G-component of $M_{(H)}$ is the preimage of a component of $M_{(H)}/G$.)

THEOREM A EQ. (BIERSTONE [3]). *If V_1, V_2 are smooth G-manifolds of the same dimension and V_1 satisfies the Bierstone Condition, $d\colon I_G^\infty(V_1, V_2) \to R_G(TV_1, TV_2)$ is a weak homotopy equivalence.*

Again this theorem has a semisimplicial version. By methods analogous to the G-trivial case we get a G-version of Theorem B.

THEOREM B EQ. *If M_1, M_2 are locally smooth G-manifolds of the same dimension and M_1 satisfies the Bierstone Condition, $d\colon I_G^t(M_1, M_2) \to R_G^t(\tau M_1, \tau M_2)$ is a weak homotopy equivalence.*

Again $I_G^t(M_1, M_2)$ and $R_G^t(\tau M_1, \tau M_2)$ are simplicial sets. Also τM is a G-microbundle; i.e., G acts on the total space through microbundle maps.

The notion of local triviality for G-vector bundles is somewhat more involved than for ordinary vector bundles: If ξ is a G-vector bundle over a completely regular G-space X, for each $x \in X$ there is a slice S_x (i.e., the orbit Gx through x has a G-neighborhood GS_x, G-equivalent to $G \times_{G_x} S_x$), such that $\xi|GS_x$ is equivalent to the G-vector bundle $1_\rho(S_x)\colon G \times_{G_x} (S_x \times R_\rho^n) \to G \times_{G_x} S_x$ (obvious projection), where R_ρ^n is an orthogonal G_x space, $\rho\colon G_x \to O_n$ a representation.

Note that since M is locally smooth τM is locally G-equivalent to a G-vector bundle and hence locally G-trivial in the above sense. One may prove a G-Kister theorem for locally G-trivial microbundles and show the category of locally G-trivial microbundles coincides with the category of locally G-trivial G-R^n bundles.

Now $T(G \times_{G_x} R_\rho^n) = G \times_{G_x} (R_\rho^n \times R_\rho^n)$ and we have an obvious G-vector bundle map of $1_\rho(S_x) \to T(G \times_{G_x} R_\rho^n)$ sending S_x to $0 \in R_\rho^n$.

Thus again we have that if τM contains a G-vector bundle ξ we can cover M by G-invariant neighborhoods $U = GS_x$ such that $\xi|U$ is G-trivial and hence we get a G-immersion $U \to G \times_{G_x} R_\rho^n$ and a G-smoothing of U by Lemma 1 eq. Then using Theorems A eq. and B eq., we get by an argument completely analogous to the G-trivial case that if M satisfies the Bierstone Condition and τM reduces to a G-vector bundle ξ, then M has a G-smoothing corresponding to the reduction of τM to ξ (cf. [2]).

To obtain a result for arbitrary G-manifolds we must use a G-engulfing theorem. This is proved from the ordinary engulfing theorem by inducing up the orbit types and leads to:

THEOREM C EQ. *If* $\dim H \neq 4$ *for any* $H \subset G$, *the isotopy classes of G-smoothings of M are in bijective correspondence with the homotopy classes of G-vector bundle reductions of* τM.

We remark that it isn't necessary to assume M is locally smooth, because it is easy to see that if τM reduces to a G-vector bundle then M must be locally smooth.

The obstructions to reducing τM to a G-vector bundle lies in $\pi_i(\mathrm{Top}_n^\rho/O_n^\rho)$, where $\rho: H \to O_n$ and Top_n^ρ (O_n^ρ) is the subgroup of Top_n (O_n) commuting with the orthogonal action of H.

Now $R_\rho^n = R_\alpha^k \oplus R^l$, $k + l = n$, where we have split off the trivial representations. Write $\mathrm{Top}_n^\rho = \mathrm{Top}_{k+l}^\alpha$ and $O_n^\rho = O_{k+l}^\alpha$. Then if we let $C^\alpha(S^{k+l})$ be the subgroup of $C(S^{k+l})$ commuting with the action of H on $I \times S^{k+l}$ (trivial action on I, orthogonal action on S^{k+l}), we again have a fibration:

$$C^\alpha(S^{k+l}) \to \mathrm{Top}_{k+l}^\alpha/O_{k+l}^\alpha \to \mathrm{Top}_{k+l+1}^\alpha/O_{k+l+1}^\alpha.$$

Here however, the groups $\pi_i C^\alpha(S^{k+l})$ are *not* zero in general. In principle, they can be computed by methods of Anderson and Hsiang [**1**]. In particular, if H acts freely on S^{h-1} via α then $\pi_i C^\alpha(S^{h+l}) \simeq \pi_i C^\alpha(S^{h+l} \bmod S^l) \oplus \pi_i C(S^l)$; and if $k + l \geqslant 6$, Anderson and Hsiang have shown:

$$\pi_i C(S^{h+l} \bmod S^l) \simeq \begin{array}{ll} K_{-l+1+i}(Z(H)), & i < l - 1 \\ \tilde{K}_0(Z(H)), & i = l - 1 \\ \mathrm{Wh}_1(H), & i = l \\ \pi_{i-l-1} C(L \times D^{l+1}), & i > l \end{array}$$

where $L = S^{h-1}/H$ and the K_{-j} are Bass' algebraic K groups.

Let M^n be a locally smooth H-manifold for which the action is semifree. Suppose $\dim M^H = l$, $n = k + l$ and $\alpha: H \to O_k$ is the representation of H on the normal disc to M^H. Then the obstructions to H-smoothing lie in $\mathrm{Top}_{k+l}^\alpha/O_{k+l}^\alpha$ and in Top_n/O_n if $\dim M^H \neq 4$ and $\dim M \neq 4$. For this we need know $\pi_i(\mathrm{Top}_{k+l}^\alpha/O_{k+l}^\alpha)$ only for $i \leqslant l$ and $\pi_i\, \mathrm{Top}_n/O_n$ for $i \leqslant n$.

Now Top_l/O_l is a retract of $\mathrm{Top}_{k+l}^\alpha/O_{k+l}^\alpha$. We also have the inclusion of $A^\alpha(S^{k-1})/O_k^\alpha \to \mathrm{Top}_{k+l}^\alpha/O_{k+l}^\alpha$, where $A^\alpha(S^{k-1})$ = group of homeomorphisms of S^{k-1} commuting with α. It can be shown that this map induces a split injection

$$\pi_i \tilde{A}^\alpha(S^{k-1})/O_k^\alpha \to \pi_i(\mathrm{Top}_{k+l}^\alpha/O_{k+l}^\alpha, \mathrm{Top}_l/O_l), \quad i \leqslant l;$$

where $\tilde{A}^\alpha(S^{k-1})$ = group of block homeomorphisms of S^{k-1} commuting with α (see [**12**]). Hence we get a split injection:

$$\pi_i\big(\tilde{A}^\alpha(S^{k-1})/O_k^\alpha\big) \oplus \pi_i(\mathrm{Top}_l/O_l) \to \pi_i \mathrm{Top}_{k+l}^\alpha/O_{k+l}^\alpha, \quad i \leqslant l.$$

Further, from the fibration above, using the fact that $\pi_i C(S^l) = 0$, $i \leqslant l + 1$, we get the exact sequence:

$$0 \to \pi_{l+1}\left(\tilde{A}^{\alpha}(S^{k-1})/O_k^{\alpha}\right) \oplus \pi_{l+1}(\mathrm{Top}_{l+1}/O_{l+1})$$

$$\to \pi_{l+1}(\mathrm{Top}_{k+l+1}^{\alpha}/O_{k+l+1}^{\alpha}) \to \mathrm{Wh}_1(H) \to \pi_l(\mathrm{Top}_{k+l}^{\alpha}/O_{k+l}^{\alpha})$$

$$\to \pi(\mathrm{Top}_{k+l+1}^{\alpha}/O_{k+l+1}^{\alpha}) \to \tilde{K}_0(Z(H)) \to \pi_{l-1}(\mathrm{Top}_{k+l}^{\alpha}/O_{k+l}^{\alpha})$$

$$\to \pi_{l-1}(\mathrm{Top}_{k+l+1}^{\alpha}/O_{k+l+1}^{\alpha}) \to K_{-l+1}\left(Z(H)\right) \to \pi_0(\mathrm{Top}_{k+l}^{\alpha}/O_{k+l}^{\alpha})$$

$$\to \pi_0(\mathrm{Top}_{k+l+1}^{\alpha}/O_{k+l+1}^{\alpha}) \to K_{-l}(Z(H)).$$

Of course, $\pi_{l+1}(\mathrm{Top}_{l+1}/O_{l+1}) \simeq \pi_{l+1}(\mathrm{Top}/O)$. Also $\pi_i(\tilde{A}^{\alpha}(S^{k-1})/O_k^{\alpha})$ can be computed up to extension from the surgery exact sequence for L.

Finally, we note the following results of Bass and others for the algebraic K-groups.

For π abelian, $K_{-j}(Z(\pi)) = 0$ for $j > 1$.

For π abelian and prime power order, $K_{-1}(Z(\pi)) = 0$.

For π cyclic of order p, $\tilde{K}_0(Z(\pi)) =$ class group of $Q(e^{2\pi i/p})$.

For π finite $\tilde{K}_0(Z(H))$ is finite.

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