BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 1, January 1978

BOOK REVIEWS

A comprehensive introduction to differential geometry, by Michael Spivak, Publish or Peril, Inc., Boston, Mass., volume 3, 1975, 474 + ix pp., \$16.25; volume 4, 1975, v + 561 pp., \$17.50; volume 5, 1975, v + 661 pp., \$18.75.

Spivak's *Comprehensive introduction* takes as its theme the classical roots of contemporary differential geometry. Spivak explains his Main Premise (my term) as follows: "in order for an introduction to differential geometry to expose the *geometric* aspect of the subject, an historical approach is necessary; there is no point in introducing the curvature tensor without explaining how it was invented and what it has to do with curvature". His second premise concerns the manner in which the historical material should be presented: "it is absurdly inefficient to eschew the modern language of manifolds, bundles, forms, etc., which was developed precisely in order to rigorize the concepts of classical differential geometry".

Here, Spivak is addressing "a dilemma which confronts anyone intent on penetrating the mysteries of differential geometry". On the one hand, the subject is an old one, dating, as we know it, from the works of Gauss and Riemann, and possessing a rich classical literature. On the other hand, the rigorous and systematic formulations in current use were established relatively recently, after topological techniques had been sufficiently well developed to provide a base for an abstract global theory; the coordinate-free geometric methods of E. Cartan were also a major source. Furthermore, the viewpoint of global structure theory now dominates the subject, whereas differential geometers were traditionally more concerned with the local study of geometric objects.

Thus it is possible and not uncommon for a modern geometric education to leave the subject's classical origins obscure. Such an approach can offer the great advantages of elegance, efficiency, and direct access to the most active areas of modern research. At the same time, it may strike the student as being frustratingly incomplete. As Spivak remarks, "ignorance of the roots of the subject has its price–no one denies that modern formulations are clear, elegant and precise; it's just that it's impossible to comprehend how any one ever thought of them."

While Spivak's impulse to mediate between the past and the present is a natural one and is by no means unique, his undertaking is remarkable for its ambitious scope. Acting on its second premise, the *Comprehensive introduction* opens with an introduction to differentiable manifolds; the remaining four volumes are devoted to a geometric odyssey which starts with Gauss and Riemann, and ends with the Gauss-Bonnet-Chern Theorem and characteristic classes. A formidable assortment of topics is included along the way, in which we may distinguish several major historical themes:

In the first place, the origins of fundamental geometric concepts are investigated carefully. As just one example, Riemannian sectional curvature is introduced by a translation and close exposition of the text of Riemann's remarkable paper, *Über die Hypothesen, welche der Geometrie zu Grunde* *liegen*. Secondly, Spivak gives extensive attention to the beautiful theorems of classical global surface theory. Such theorems offer an intuitively appealing introduction to the modern viewpoint in differential geometry, a fact which has also been recognized by the various excellent undergraduate textbooks which are now available. Thirdly, some currently unfashionable topics are included. For example, there is a treatment of affine surface theory, which can serve as an introduction to Cartan's approach to differential invariants. There is also a highly selective course on partial differential equations for geometers, including a study of the Darboux equation and of the Cartan-Kähler theory of differential systems.

The *Comprehensive introduction* is probably best suited for leisurely and enjoyable background reference by almost anyone interested in differential geometry. Great care has been taken to make it accessible to beginners, but even the most seasoned reader will find stimulating reading here (including instances of good work forgotten and recently redone). The appeal of the book is due first of all to its choice of material, which is guided by the liveliest geometric curiosity. In addition, Spivak has a clear, natural and well-motivated style of exposition; in many places, his book unfolds like a novel.

A warning may be in order, however, to take the Main Premise with a grain of salt. The fact is that Spivak's explanations are sometimes too thorough to make good *introductory* reading. For instance, Volume 2 contains *seven* proofs that the vanishing of the Riemannian curvature tensor implies the existence of a local isometry with Euclidean space. These proofs are distributed throughout the discussion of formalisms for the notion of covariant derivative, or connection, and illustrate the strengths of the various formalisms as computational tools. Thus the first proof is a long but straightforward computation, in which the curvature tensor arises as it did historically and the last is a triumph of brevity set in an elaborate framework. Surely, such careful accounts as this are better suited to a reader who has already had some encounter with modern differential geometry, and is therefore sufficiently confused to appreciate them.

Later on we shall offer some further comments about the book as a whole. However, since each volume has its own character, it will be helpful to consider the volumes separately. The first two were reviewed previously by Guillemin (Bull. Amer. Math. Soc. **79** (1973), 303–306), but are briefly included here for completeness.

VOLUME 1. DIFFERENTIABLE MANIFOLDS. As we have already mentioned, much of this volume is devoted to basic material about manifolds, differential equations on manifolds, and differential forms. The account is distinguished by its elementary prerequisites, specifically, advanced calculus and a basic knowledge of metric spaces, and by its careful attention to motivation. It is also a lively account, full of examples, excellent informal drawings which function as part of the text, and stimulating problem sets. (The problems give out almost entirely after this volume, but the examples and drawings persist.)

The Main Premise comes into play in several places. For instance, Spivak treats integral submanifolds in terms of classical integrability conditions, before reformulating in terms of vector fields or differential forms. Thus he initiates one of the book's minor themes, namely, the "incredibly concise and elegant" disguises assumed by integrability conditions in differential geometry. In another instance, from a later chapter on Riemannian metrics, the geodesic equation is derived from the classical calculus of variations and Euler's equation, without introducing the notion of connection; this makes worthwhile supplementary reading for a standard presentation of geodesics.

A particularly good feature of this volume is its treatment of algebraic topology from the differentiable viewpoint. By restating algebraic-topological theorems in terms of the de Rham cohomology (a cohomology theory defined in terms of differential forms), Spivak is able to achieve significant simplifications in exposition while still conveying much of the flavor of the subject. (He also provides for the needs of the final volume, by discussing the Thom cohomology class and the equivalence of various differential-topological definitions of the Euler characteristic.)

VOLUME 2. GAUSS AND RIEMANN. CONNECTIONS. This year is the one hundred and fiftieth anniversary of Gauss' famous treatise on surfaces in R^3, *Disquisitiones generales circa superficies curvas*. In this fundamental paper, Gauss established the Theorema Egregium and the angular defect theorem for geodesic triangles, and made the first systematic use of the local parametrization of surfaces by two variables. In 1854, Riemann followed with the concept of an "n-fold extended quantity" (now a differentiable manifold), susceptible to various quadratic metric structures; he justified his program to free differential geometry from its three-dimensional Euclidean framework by arguing for non-Euclidean conceptions of Space.

These works are of tremendous historical interest, and Spivak must be thanked for his illuminating exposition of them. Riemann's paper presents particular difficulties, omitting almost all computations and greatly exceeding the bounds of the mathematical language of the time. Spivak provides thirty-five pages of computation to back up Riemann's nine-sentence derivation of sectional curvature and its properties from the Taylor expansion in normal coordinates of the metric. (Actually, the ninth sentence is not explained fully until later. Its claim that the curvature determines the metric depended on a "counting argument", and was only proved rigorously many years afterward, as the "Cartan local isometry theorem".) It is disappointing, if understandable, that Spivak draws the line at mathematics, and does not take up the subject of Riemann as a prophetic physicist.

Most of the rest of the volume is devoted to developing and comparing connection formalisms. Students of differential geometry are generally expected to become proficient in these tools of the trade gradually and by assimilation. The disadvantages of being completely explicit are apparent in the present treatment, which is somewhat pedantic. Nonetheless, the detailed tabulation which it provides will be a valuable aid to the assimilation process.

VOLUME 3. SURFACE THEORY IN THE LARGE. The theorems of classical global surface theory have great geometric appeal, and lie at the roots of much current research. Such an example is the Gauss-Bonnet Theorem, which relates the integral of Gaussian curvature over a compact surface to a purely topological invariant, the Euler characteristic; its modern reincarnation, which Spivak treats in Volume 5, represents one of the major achievements of modern mathematical machinery. Other examples include

Hilbert's theorem that there are no complete immersed surfaces of constant negative curvature in R^3, and Hadamard's theorem that a compact immersed surface of positive curvature in R^3 bounds a convex body. By contrast, the major recent work associated with these, due to Efimov and Sacksteder, respectively, has depended upon great ingenuity and little machinery.

An excellent selection of fundamental theorems on surfaces is the main subject of this third volume. Some modern work is included, namely, Kuiper's theorem on surfaces of minimal total absolute curvature, as well as work of Hartman and Nirenberg, Massey, and Maltz related to the cylinder theorem for surfaces of vanishing curvature. In addition, there is a systematic and well illustrated compendium of examples.

VOLUME 4. VARIATION THEORY. RIEMANNIAN SUBMANIFOLDS. In this volume, Spivak moves into higher dimensions, and continues his exposition of the roots of contemporary global geometry. For example, outgrowths of the second variation formula for arclength, especially the Rauch Comparison Theorem and the Toponogov Triangle Theorem, are major tools of the beautiful modern theorems which relate topology to curvature through comparison with constantly curved model spaces. Such standard reference works as those by Kobayashi and Nomizu, Bishop and Crittenden, or Milnor (*Morse theory*) devote considerable attention to second variation of arclength, and the treatise by Cheeger and Ebin covers recent work in detail. Spivak provides an introduction to the area by a careful exposition of the Rauch Comparison Theorem, as well as theorems of Synge and Klingenberg.

On the other hand, his unusually extensive chapter on Riemannian submanifolds goes beyond being a good exposition of readily available material, and performs a scholarly service. For instance, it pulls together "leftover problems from classical differential geometry" by tabulating known results and remaining open questions about complete surfaces of constant curvature in Euclidean, spherical or hyperbolic space. A strange fact in the history of surface theory is that it was only recently discovered that complete surfaces in R^3 with vanishing curvature are cylinders. Spivak produces another surprise, namely, that the complete surfaces in S^3 with vanishing curvature were elegantly treated by Bianchi in 1896. Another valuable inclusion is a modern treatment of a geometrically appealing, although involved, theory of submanifold invariants due to Burstin, Mayer and Allendoerfer. This theory, which involves higher-order osculating spaces and higher-order normal connections, has not been at all well known, and might well have been independently reworked if Spivak had not called attention to it.

VOLUME 5. (PART 1). PDE. RIGIDITY. Many of the standard theorems about partial differential equations have applications to the geometric theory of imbedding and rigidity. For instance, a negatively curved surface in R^3 can locally be *bent* continuously along an asymptotic curve, or *warped* into exactly one other position along a nowhere asymptotic curve. (Here, the rigidity terminology is Spivak's, who points out the ambiguous state of the current terminology in English.) This geometric theorem reduces to the Cauchy problem for the Darboux equation, a nonlinear second order equation of Monge-Ampère type, with hyperbolic initial data. Or again, the analyticity of surfaces which realize analytic positively curved metrics follows from the analyticity of solutions of elliptic type for analytic second order equations. As another example, the Burstin-Janet-Cartan Theorem, which states that any analytic n-dimensional metric is realizable locally by an imbedding in Euclidean space of dimension $\frac{1}{2}n(n+1)$, may be reduced to the Cauchy-Kowalewski Theorem.

Spivak has prepared a course on PDE for geometers, with an eye to these and similar applications. We have already mentioned that his course is selective; it omits such major topics as the Dirichlet problem. On the other hand, its chosen topics are treated rigorously and in considerable generality. As always, Spivak emphasizes conceptually appealing proofs. For instance, the analyticity of solutions of nonlinear, second order elliptic problems in two variables is proved by Lewy's elegant method, which uses the hyperbolic initial value problem to extend solutions into the complex domain and there to verify the Cauchy-Riemann equations.

There is also a highly enjoyable general discussion of rigidity, an area which is characterized by ingenious proofs and intriguing open questions. The most famous rigidity theorem is Cohn-Vossen's, which states that every compact convex surface is unwarpable. This and other classical uniqueness theorems for convex surfaces may be viewed as uniqueness theorems for (possibly nonlinear) boundary value problems in PDE, and their proofs may be expected to depend upon the invention of special global devices; Spivak uses integral formulas due to Blaschke, Herglotz and Chern. He does not go into the powerful and intricate methods of the Soviet school, which grew out of Alexandrov's work of the 1940's and has virtually no practitioners writing in English. However, an account is included of an "incredibly neat trick" due to Pogorelov for transferring rigidity theorems in R^3, to S^3 or H^3.

VOLUME 5. (PART 2). CHARACTERISTIC CLASSES. The book's final chapter is entitled "The Generalized Gauss-Bonnet Theorem and What It Means for Mankind". One of its meanings, as Spivak undertakes to illustrate, is that continued scrutiny of one of the most beautiful theorems of classical differential geometry has led inevitably into the spirit of functorial construction.

The Generalized Gauss-Bonnet Theorem, first proved by Chern, states that for a compact, oriented Riemannian manifold of fixed even dimension n, the integral of the Pfaffian of curvature is proportional by a universal constant to the Euler characteristic. It may be derived from certain basic facts which include the following: First, that for oriented real n-plane bundles E over compact oriented base manifolds, both the Euler class and the Pfaffian of curvature (with respect to any metric connection on E) are natural ways to obtain an n-dimensional cohomology class on the base manifold. Here, *natural* means commuting with pull-back. Secondly, that such a bundle E is the pull-back of a "universal" bundle, with base manifold the oriented Grassmann manifold $G(n, N)$ for N sufficiently large.

Spivak first presents this "proof by magic"; the remainder of the chapter he devotes to revealing some of the mechanisms behind it. What is at issue is to understand the natural, or "characteristic", classes of oriented real n-plane bundles. These classes are the basic invariants which measure the deviation of a bundle's local product structure from being a global product structure. Spivak devotes sixty pages to developing the relationship between characteristic classes and curvature in such a way that the Weil homomorphism is seen to appear naturally. His computations of the de Rham cohomology of $G(n, N)$ are carried out by purely differential geometric methods, using the identification of $G(n, N)$ with a quotient of Lie groups and making no appeal to the advanced machinery of algebraic topology. Several of the book's outstanding virtues are represented in this treatment: it is self-contained; it gives more than cursory attention to classical invariant theory; and it prizes and imparts geometric insight.

After such a detailed discussion of the good things in the *Comprehensive introduction*, perhaps we should also look briefly for flaws. They are of the sort that would be expected in a work of such magnitude written over a relatively short period of time. As Spivak says, "what I have written is a second or third draft of a preliminary version". Indeed, there is evidence that he originally expected to write only two volumes, and that the book simply took over. Thus one can find occasional instances of loose organization, sketchy referencing, and oversight. (The first two volumes have been carefully corrected in the separate Errata given in Volumes 2, 3 and 5; especially out of consideration for graduate students, it might be good to publish the corrections to the later volumes also.) However, these things are minor, and do not detract from the pleasure of the book. Perhaps more importantly, some readers may be disappointed by a certain lack of synthesis, and wish that Spivak had revealed, for the sake of argument at least, what conclusions he has drawn about differential geometry, its history, and its future.

But it would be ungrateful to ask for more than Spivak has already given us. The *Comprehensive introduction* will be widely read and enjoyed, and will surely become a standard reference for graduate courses in differential geometry. Spivak is greatly to be thanked for this spontaneous, exuberant and beautifully geometrical book.

STEPHANIE ALEXANDER

BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 1, January 1978

Modular representations of finite groups, by B. M. Puttaswamaiah and John D. Dixon, Academic Press, New York, San Francisco, London, 1977, vii + 242 pp. $23.50.

The classical theory of the complex representations of a finite group G can be studied in a number of different but closely related ways. First, one can work with the actual representations, the homomorphisms of G into complex general linear groups. This leads to complex valued functions on G, in particular, the characters, and a very function oriented approach. Another way is the study of the ring theoretic structure of the complex group algebra which leads to idempotents, ideals, left ideals and so on. Finally there is the module approach and the study of homomorphisms, endomorphisms, tensor products. These three different methods exist side by side. For example, there