Measures on topological semigroups: Convolution products and random walks,
by Arunava Mukherjea and Nicholas A. Tserpes, Lecture Notes in Math.,
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$10.20.

This set of notes concerns itself with a selection of topics concerning
measures and convolution of measures on topological semigroups. A topo-
logical semigroup as treated here is simply a semigroup with a topology on it
so that the binary multiplicative operation is jointly continuous with respect
to the topology. A number of the directions pursued and results discussed are
motivated by an existing theory for topological groups and an attempt to
generalize it. The classical result on the existence of left and right invariant
(Haar) measures for locally compact topological groups represents such an
example. Let S be a locally compact topological semigroup. The following
notions of right invariance have been introduced for nonnegative regular
Borel measures on S. μ is called r^*-invariant if $\mu(Bx^{-1}) = \mu(B)$ for each
$x \in S$ and $B \in \mathfrak{B}$ (the σ-algebra of Borel sets). Here $Bx^{-1} = \{y: yx \in B\}$.
μ is called right invariant if $\mu(Kx) = \mu(K)$ for each compact $K \subset S$. A
semigroup is called a *left group* if $Sx = S$ for all $x \in S$ and it is right
cancellative. The authors show that if a probability measure μ is r^*-invariant
then the support of the measure S_μ is a left group. Further the support S_μ can
be represented as the product $E \times G$ of a locally compact left zero semigroup
($ee' = e$ for e, $e' \in E$) and a compact group G, and the measure μ as a
product measure $\mu_1 \times \mu_2$ with μ_1 a probability measure on E and μ_2 normal-
ized Haar measure on the group G. Only a limited version of this result has
been obtained for infinite measures using the notion of both r^*-invariant and
right invariant measure.

Many of the problems dealt with are suggested by probabilistic questions
and the results obtained are a curious interplay of ideas from probability
theory, functional analysis and measure theory. It is helpful to introduce the
following characterization of a completely simple semigroup. A semigroup S
is completely simple if and only if there are sets X and Y, a group G, a
function $\phi: Y \times X \to G$ such that S is isomorphic to the semigroup $X \times G$
$\times Y$ with multiplication defined by

$$(x_1, g_1, y_1)(x_2, g_2, y_2) = (x_1, g_1\phi(y_1, x_2) g_2, y_2).$$

Most of the results on compact Hausdorff semigroups are rather complete
and much of the research of the authors is concerned with an effort to extend
these results to, for example, locally compact semigroups. If S is a compact
Hausdorff semigroup, it has a nonempty minimal two-sided ideal K (or
kernel). Further the kernel K is closed and completely simple.

Given any two regular probability measures μ, ν on a locally compact
topological semigroup, the convolution $\eta = \mu * \nu$ of μ and ν is introduced as
follows. Consider the iterated integral

$$I(f) = \int \int f(ss') \, \mu(ds)\nu(ds'),$$

for any continuous function f with compact support. By the Riesz representation theorem there is a regular probability measure η (the convolution) such that $I(f) = \int f(s)\eta(ds)$. One of the principal themes in these lecture notes is the study of the asymptotic behavior of the convolution sequence μ, $\mu^{(2)} = \mu * \mu, \ldots, \mu^{(n)}, \ldots$ generated by the regular probability measure μ as $n \to \infty$. One can without any loss of generality assume that the topological semigroup S is generated by the support S_μ of μ, that is, assume that S is the closure of $\bigcup_{k=1}^{\infty} S_\mu^k$. Again, the results are fairly complete and clear in the case of a compact Hausdorff semigroup. In that case the mass on $\mu^{(n)}$ concentrates on the kernel K of S as $n \to \infty$, that is, for any open set O containing K, $\mu^{(n)}(0) \to 1$ as $n \to \infty$. Also one has weak-star convergence of $(1/n)\sum_{j=1}^{n} \mu^{(j)}$ as $n \to \infty$ to a probability measure η that is idempotent, $\eta^{(2)} = \eta$. For this reason, it is of some interest to characterize the form of an idempotent probability measure η. If S is a locally compact semigroup that is the support of a regular idempotent probability measure η, then S must be completely simple and in the standard representation of S, $X \times G \times Y$, the group factor G must be compact. Further η decomposes on $X \times G \times Y$ as a product measure $\eta = \eta_1 \times \eta_2 \times \eta_3$ with η_1, η_3 as probability measures on X, Y respectively and η_2 normalized Haar measure on G. Again assume that S is a compact Hausdorff semigroup and generated by the support of the probability measure μ. If the convolution sequence $\mu^{(j)}$ converges as $j \to \infty$, the limit must be an idempotent measure η. Let $X \times G \times Y$ be the standard representation of the kernel of K. Then the sequence $\mu^{(j)}$ will not converge if and only if there is a proper closed normal subgroup G' of G such that $YX \subset G'$ and $(X \times G' \times Y)S_\mu \subset X \times gG' \times Y$ with $g \in G$ but $g \notin G'$. On the other hand, if S is a locally compact second countable completely simple semigroup and is generated by the support of a probability measure on S, then $\mu^{(n)} \to 0$ vaguely as $n \to \infty$ if and only if the group factor G in the standard representation $X \times G \times Y$ of S is not compact. This sort of interesting result in the noncompact case gives a perspective on the limited insight one has on limit behavior in this more difficult context. There are also results on the convergence of convolutions of nonidentically distributed measures. A treatment of many of these questions can also be found in [7].

There is a detailed discussion of the problem of characterizing features of the idempotent limit measure in some special but interesting cases. Even in the case of the semigroup of 2×2 transition probability matrices, with μ a two point measure it is not easy to determine, for example, whether the corresponding idempotent measure is absolutely continuous or singular. Results of the authors and their coworkers in this direction are presented.

The problems dealt with can be given a direct probabilistic interpretation using the formalism of Markov processes or random walks. Since

$$\mu^{(2)}(B) = \int \mu(Bx^{-1})\,\mu(dx) = \int \mu(x^{-1}B)\,\mu(dx),$$

where $x^{-1}B = \{y: xy \in B\}$, one can associate with the probability measure μ right and left Markov processes or random walks with stationary transition probabilities $P_r(x, B) = \mu(x^{-1}B)$, $P_l(x, B) = \mu(Bx^{-1})$ respectively. A bilateral random walk on S with transition probability $P_b(x, B) =$

$\mu * \delta_x * \mu(B)$ can also be associated with μ. The right random walk can be interpreted as follows. Let $X_1, X_2, \ldots$ be independent, identically distributed random variables taking on values in S and having common distribution μ. The right random walk is the sequence of products $Z_n = X_1 X_2 \cdots X_n$, $n = 1, 2, \ldots$. A right random walk on a topological group G is said to be recurrent if there is a value (recurrent) $x \in G$ such that

$$P(Z_n \in N(x) \text{ infinitely often}) = 1,$$

for every neighborhood $N(x)$ of x. x is a possible value of the random walk if for each neighborhood $N(x)$ of x there is an integer $n > 0$ such that $P(Z_n \in N(x)) > 0$. Chung and Fuchs [1] showed that in the case of Euclidean k-space there are either no recurrent values or else all possible values are recurrent. Further, the random walk is recurrent if and only if

$$\int_V \frac{du}{1 - \phi(\mu)} = \infty,$$

for some compact neighborhood V of the identity, with ϕ the Fourier (Stieltjes) transform of the measure μ generating the random walk. Kesten and Spitzer [3] obtained analogous results for countable locally compact Abelian groups. These results were extended to the case of general locally compact Abelian groups by Port and Stone [6]. There is still the interesting question on criteria for recurrence on noncommutative noncompact groups. The notion of recurrence has to be modified for random walks on semigroups. A point x is recurrent for a right random walk if

$$P(Z_n \in N(x) \text{ infinitely often}, n > 0 \mid Z_0 = x) = 1,$$

for all neighborhoods $N(x)$ of x. The notion of recurrence for left and bilateral random walks is analogous. The notions of recurrence are shown to be equivalent for unilateral and bilateral random walks when the semigroup is compact. Again there are a number of open problems in the case of noncompact random walks. The discussion of recurrence for random walks on semigroups is motivated in part by work of Larisse [4] and Högnas [2]. There has been a great deal of research on random motions on noncommutative groups (see for example [5]) in recent years that is not dealt with in this monograph. Also see [7] for an application of related work to a one-sided representation theorem for Markov sequences in terms of independent random variables.

All in all, the lecture notes of Mukherjea and Tserpes provide an interesting and well-motivated introduction to problems concerning measures on topological semigroups and the random walks generated by these measures. The little monograph is a pleasant addition to the Springer series of lecture notes in mathematics.

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Adventures of a mathematician, by S. M. Ulam, Charles Scribner's Sons, New York, 1976, xi + 317 pp., $14.95.

I. Ulam is a magic name in modern mathematics. One thinks of Leonardo's letter to the Duke of Milan:

"Most Illustrious Lord;

. . . Item: In case of need I will make big guns, mortars, and light ordnance of fine and useful forms, out of the common type.

Item: I can carry out sculpture in marble, bronze, or clay, and also I can do in painting whatever may be done, as well as any other, be he who he may . . . ."

And so he could.

In Ulam's writing, as in Leonardo's, scarcely a mention of mother and father. At eleven Ulam began to be known as a bright child who understood the special theory of relativity. He was an A student but did not study much, active in sports, played bridge, poker, and chess. At 15 he absorbed the calculus, number theory, and set theory. At 18, when he matriculated from gymnasium, the choice of profession presented difficulties. His father wanted him to join his successful law practice, while Ulam longed for a university career. But university positions in Poland were almost impossible to obtain if one's family, however wealthy and culturally assimilated, had a Jewish background. As a compromise, Ulam entered Lwów, Polytechnic Institute to study engineering.

From the first, mathematics took complete possession of him. Kuratowski quickly recognized the young student's gifts and took special pains with him. The names of Mazur, Lomnicki, Borsuk, Kacmarz, Nikliborc, Tarski, Schauder, Averbach, Schreier, Steinhaus, and above all Banach dominated a euphoric period of feverish activity. At 23 Ulam was sufficiently well known to be an invited speaker at the Zürich congress. Meeting foreign mathematicians for the first time, he found them nervous and given to facial twitches, or short and old, like Hilbert; certainly less impressive than his fellow Poles. Returning to Lwów, Ulam wrote a master's thesis which among other things outlined what is now category theory, and at 24 won his doctorate with a thesis in measure theory. But still there were no prospects of a university position for him in Poland.