turn out idiot savants in the sciences as being more likely to be useful to the state. But if mathematical intelligence is strongly associated with emotional deprivation and social alienation, then even we earthy, super-honest, solid, and simple native Americans–the qualities that Ulam admires in us–are in for trouble.

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Interpolation spaces, an introduction, by Jöran Bergh and Jörgen Löfström, Springer-Verlag, Berlin, Heidelberg, New York, 1976, x + 204 pp., $24.60.

We don't want to get involved in the current, often heated, debate as to what constitutes pure, as opposed to applied, mathematics. We just want to start this review by saying that the theory of interpolation of operators is an impressive application of pure mathematics to pure mathematics. The purists (?) are welcome to wrangle over the semantics.

The subject has its origins in classical Fourier analysis, where it was conceived as an elementary means of finding L^p-estimates. The very nature of interpolation theory, however, is functional-analytic: typically, a linear operator T is bounded between spaces X_α and Y_α when $\alpha = 0$ and $\alpha = 1$, and one wants to conclude that T carries X_α to Y_α whenever $0 < \alpha < 1$. Such problems arise in many areas of analysis, and the abstract theory has always been influenced, even guided, by the potential applications to such areas as harmonic analysis, approximation theory, and the theory of partial differential equations. As a result, interpolation has no one place to call home; it is, quite simply, interesting mathematics.

Its success, like that of a good executive, stems from its ability to handle specifics while operating on a generally higher plane. Consider, for example, the thorny Fourier and Hilbert transforms. A great deal of highly-specialized information is known about these operators, and it rarely comes for free. Yet, remarkably enough, the "correct" L^p-estimates can be derived from general interpolation theorems valid for *all* linear operators.

Such examples show that it is worthwhile to solve the interpolation problem simultaneously for all operators. It also changes the face of the problem, because the operators themselves, since only their linearity is important, tend to fade into the background. Interpolation theorems then are more properly construed as statements about the underlying system of spaces. This observation, simple as it is, represents the point of departure from classical L^p-interpolation (the Riesz-Thorin and Marcinkiewicz theorems) into the abstract theory of interpolation spaces and interpolation methods.

Suppose $\{X_\alpha: 0 \leqslant \alpha \leqslant 1\}$ is a family of Banach spaces for which an interpolation theorem is desired. The idea is to construct, using only the extremal spaces X_0 and X_1, an intermediate space $(X_0, X_1)_\alpha$, say, for which the interpolation property *automatically* holds. Several of these constructions, called *interpolation methods*, are known. What remains, and this is often the hard part, is to identify the *interpolation space* $(X_0, X_1)_\alpha$ with the original space X_α.

All of the known constructions are manifestations of one or other of the two basic interpolation methods, the *complex method* and the *real method.* These methods in turn are based on the main ideas underlying the proofs of the Riesz-Thorin and Marcinkiewicz theorems, respectively. The complex and real methods themselves are inequivalent and frequently produce different families of interpolation spaces.

The starting-point is a pair of Banach spaces X_0 and X_1 (more general kinds of spaces are also allowed), each of which is continuously contained in the same Hausdorff topological vector space $\mathfrak{X}$. The intersection $X_0 \cap X_1$ and the sum $X_0 + X_1$ can then be formed inside $\mathfrak{X}$. They are Banach spaces under the norms

$$\|x\|_{X_0 \cap X_1} = \max(\|x\|_{X_0}, \|x\|_{X_1}); \qquad \|x\|_{X_0+X_1} = \inf(\|x_0\|_{X_0} + \|x_1\|_{X_1}),$$

where the infimum is taken over all representations $x = x_0 + x_1$, with $x_0 \in X_0$ and $x_1 \in X_1$. The pair (X_0, X_1) is called a *compatible couple.*

In the complex method, devised independently by A. P. Calderón, S. G. Krein and J. L. Lions in about 1960, analyticity is used to select the interpolation spaces. Let F be a bounded, continuous function on the strip $\{0 \leqslant \operatorname{Re} z \leqslant 1\}$, taking values in $X_0 + X_1$, and analytic in the interior of the strip. It is prescribed that F take values in X_0 on the line $\{\operatorname{Re} z = 0\}$, and values in X_1 on the line $\{\operatorname{Re} z = 1\}$. Now suppose θ is fixed, with $0 \leqslant \theta \leqslant 1$. As F varies, vectors $x = F(\theta)$ are selected from $X_0 + X_1$, and they constitute the interpolation space $(X_0, X_1)_\theta$. It is a Banach space under the norm

$$\|x\|_{(X_0,X_1)_\theta} = \inf_{x=F(\theta)} \|F(\theta)\|_{X_0+X_1}.$$

The interpolation property follows, as in the Riesz-Thorin theorem, from the Hadamard "three-lines theorem".

The real method was introduced by E. Gagliardo in 1959. Equivalent, but far simpler, methods were developed by J. L. Lions, E. T. Oklander and J. Peetre. The most natural formulation is the one obtained by Peetre. For each x in $X_0 + X_1$, and each $t > 0$, let

$$K(t; x) = \inf_{x=x_0+x_1} (\|x_0\|_{X_0} + t\|x_1\|_{X_1}).$$

For each fixed x, the "Peetre K-functional" is an increasing, concave, function of t. The interpolation spaces are selected by imposing growth restrictions on the K-functionals. Thus, for example, when $0 < \theta < 1$ and $1 \leqslant q \leqslant \infty$, the space $(X_0, X_1)_{\theta,q}$ consists of the vectors x in $X_0 + X_1$ for which the norm

$$\|x\|_{(X_0,X_1)_{\theta,q}} = \left(\int_0^\infty \left[t^{-\theta}K(t; x)\right]^q \frac{dt}{t}\right)^{1/q}$$

is finite. The proof of the interpolation property is elementary.

These are the definitions. Before they can be satisfactorily launched at the applications, a number of results of a general nature have to be established. That the true depths of the subject are indeed being plumbed here is perhaps indicated by the fact that essentially everything that one wants to work does. For instance, under minimal hypotheses, the duals of the interpolation spaces between X_0 and X_1 turn out to be the interpolation spaces between the duals X_0^* and X_1^*.

It is clear that the success of the theory will hinge on the possibility of identifying the abstract interpolation spaces with those occurring in practice. In the real method, for example, this is tantamount to obtaining an accurate description of the K-functional. Several important cases are known.

For the pair (L^1, L^∞), the K-functional is given by $K(t; f) = \int_0^t f^*(s)\, ds$, where f^* is the decreasing rearrangement of the function f. The interpolation spaces $(L^1, L^\infty)_{\theta,q}$ are then easily identified as the Lorentz spaces L^{pq}, where $\theta = 1 - 1/p$. Similar results hold for pairs (L^{pq}, L^{rs}) of Lorentz spaces; the Marcinkiewicz theorem is an easy corollary. More importantly, this description of the K-functional shows that the *natural* class of interpolation spaces between L^1 and L^∞ is the class of rearrangement-invariant Banach function spaces. This forges the essential link between the real method and A. P. Calderón's fundamental theory of interpolation on rearrangement-invariant spaces; the two theories are thus easily shown to be equivalent.

In the Fefferman-Stein real-variable theory of H^p-spaces, the space of functions of bounded mean oscillation (BMO) often arises as a natural substitute for L^∞. The K-functionals for the pairs (L^1, L^∞) and (L^1, BMO) are closely related. For the latter pair, it is equivalent to $\int_0^t (f^\#)^*(s)\, ds$, where $f^\#$ is the "sharp-function" in terms of which BMO is defined. This, and related results, lead to a number of basic interpolation theorems for H^p-spaces.

Let C be the space of continuous functions on the unit circle, C^1 the space of continuously-differentiable functions. The K-functional for the pair (C, C^1) is $\frac{1}{2}\omega^*(2t; f)$, where $\omega^*(t; f)$ is the least concave majorant of the modulus of continuity $\omega(t; f)$. The interpolation spaces are Lipschitz spaces $\mathrm{Lip}(\alpha; q)$ (of which the spaces $\mathrm{Lip}(\alpha; \infty)$ are the classical Lipschitz spaces $\mathrm{Lip}(\alpha)$).

When C is replaced by L^p, the counterpart of C^1 is the Sobolev space W_p^1, which consists of those functions in L^p whose first-order distributional derivatives are also in L^p. The K-functional for the pair (L^p, W_p^1) is equivalent to the L^p-modulus of continuity $\omega_p(t; f)$. The resulting interpolation spaces are the Besov spaces B_{pq}^s. The complex method produces the quite different family of "generalized Sobolev spaces" H_p^s; these are sometimes known as the "spaces of Bessel potentials". Some of the deepest applications of interpolation-space theory, involving a delicate interplay between the real and complex methods, are to be found here.

The theory of semigroups of operators produces some important generalizations of the Besov spaces. Let $\{T(t)\colon 0 \leqslant t < \infty\}$ be an equibounded, strongly-continuous, semigroup of operators on a Banach space X. Let Λ be its infinitesimal generator and let $D(\Lambda)$ be the domain of definition of Λ in X. The interpolation spaces for the pair $(X, D(\Lambda))$ are the "generalized Lipschitz spaces" $\mathrm{Lip}(\alpha, q; X)$. They reduce to the Besov spaces B_{pq}^α when $X = L^p$ and $\{T(t)\}$ is the semigroup of translations. Whatever the semigroup, these spaces measure, in a precise way, the rate of convergence of $T(t)$ to the identity, as t tends to zero. Interpolation-space theory provides a tangible means of comparing the spaces generated by different semigroups, and thus leads to a number of poweful approximation theorems. These methods have been extensively studied in an earlier text in this series (P. L. Butzer; H. Berens, *Semi-groups of operators and approximation*, Band 145).

There is another fundamental connection between the real method and approximation theory. The K-functional for the pair (X_0, X_1) is closely related to the "degree-of-approximation functional"

$$E(t; x) = \inf \|x - x_0\|_1,$$

where $x \in X_0 + X_1$, and the infimum is taken over all $x_0 \in X_0$ for which $\|x_0\|_{X_0} \leqslant t$. Comparison of the spaces generated by the K- and E-functionals leads to, for example, "Jackson- and Bernstein-type" approximation theorems, and to results on the approximation of compact operators by those of finite rank.

Given the broad nature of the subject, it is doubtful that any one text could meticulously sweep every corner. *Interpolation spaces* does, however, present a thorough treatment of the functional-analytic aspects of the theory. It is about evenly divided between theory and applications. There are some significant omissions (H^p-spaces and rearrangement-invariant spaces, for example, are barely mentioned) but this is perhaps not important, given the introductory nature of the text. One feels, however, that more background material might have been given in order to properly set the scene.

The first chapter, for instance, contains only a skeletal treatment of the Riesz-Thorin and Marcinkiewicz theorems, and provides only shallow motivation for the general theory. It also contains a number of errors, minor in themselves, but which could add up to a headache for a beginner. The assertion at the foot of p. 7 is false, and this invalidates a proof on p. 8. The parameter range in the statement of Paley's theorem is incorrect (the theorem is false for $p = 1$); the corresponding range in the statement of the Marcinkiewicz theorem is not specified at all. In the proof of the Marcinkiewicz theorem, a factor of t^{-1} is incorrectly included in the expressions for the L^p-norms, and there are a couple of typographical errors. In the proof of the Riesz-Thorin theorem (and again in Chapter 5), the authors refer to "bounded functions of compact support" when there is no apparent topology on the underlying measure spaces; even if there were, the simplistic elegance of Thorin's proof demands the use of simple functions rather than bounded functions of compact support.

This rather dismal opening should not, however, be allowed to obscure the real contributions that are made in the subsequent chapters. The strength of the book lies in its elegant treatment of the theory of interpolation spaces and interpolation methods. Here, the authors have given us a beautiful account of some beautiful mathematics. It is for this that the book will be valued and, more importantly, used.

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