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*Companion to concrete mathematics*, Volume I: *Mathematical techniques and various applications*, Volume II: *Mathematical ideas, modeling and applications*, by Z. A. Melzak, John Wiley & Sons, Inc., New York, 1973, xiii + 270 pp., \$ 14.45, 1976, xvi + 413 pp., \$29.95.

When these books (I and II) were first publicized, I has looked forward to obtaining them. Subsequently, after having skimmed through them, I felt that they were a very worthwhile addition to the literature especially in "Problem Solving". For a charming description of mathematicians as problem-solvers or theory-creators, which should give the right perspective in viewing these books, see Halmos [1]. As I more or less expected, there was much material and "tricks" which I was quite familiar with as a problem-solver. However, as a bonus there were variations and references on some of these which were new to me. Also, there was quite a bit of other material which was new to me. Consequently, at this time, I essentially agreed with a complimentary capsule review [2] of I which described it as "A delightful miscellany of problems". Additionally, I was very glad to get the publisher to denote copies of I to the top eight winners of the Fifth U.S.A. Mathematical Olympiad of 1976 (an annual high school competition of which I am chairman of the examination committee). One of the winners, a bright sophomore, wrote back to me that he found the book very interesting. Also, recently, J. Wiley has presented copies of II to the winners of the Sixth U.S.A. Olympiad (Springer-Verlag presented copies of G. Pólya, G. Szegő, *Problems and theorems in analysis*. I, II, books which I will be contrasting with the present ones, subsequently).

When I was asked to review these books, I agreed to do so, primarily, since I would then have to go through them much more thoroughly and hopefully would come up with some more interesting mathematical results which were new to me. I consider any day well spent, mathematically, if I come across even one result, proof or conjecture, mine or others, which is elegant to me. After having read the books more thoroughly, I did find more elegant results, proofs and conjectures. The range and number of topics and problems treated is very broad and large. Unfortunately, this has a dual effect akin to the Heisenberg Uncertainty Principle. On one hand, there is more than enough material to be of interest to different classes of readers; relative beginners, mature practitioners, teachers and students, pure mathematicians as well as the applied ones, and for different purposes. On the other hand, consequently, the treatment leaves much to be desired. It is often too sketchy as well as very uneven in regards to the level of difficulty. In the prefaces, the author gives his reasons for writing the books (too much abstraction, not enough geometry, too little problem formulation, lack of intuitive appeal, etc., in present-day mathematics teaching) with which this reviewer certainly agrees (see [3], [4] and the references within). He also gives reasons for his type of treatment and I quote (this also provides a brief description of the books):

"It was then that the final character of this book suggested itself to me: a collection of some body of ordinary but attractive mathematics which would supplement standard courses and texts by stressing concreteness, formal

manipulation, intuitive appeal, and ingenuity, by using physical analogies, encouraging problem formulation, and supplying problem solving methods. The material was then enriched by adding sketchy introductions to such mild esoterica as integral geometry, asymptotic analysis, Liouville's theory of the complexity of elementary functions, etc., and by inserting some brief historical references. I believe the fragmentation process is so far gone in mathematics that it may be good to sacrifice much, even in rigor, for the sake of anything which unifies seemingly distant or dissimilar subjects. This might perhaps explain some odd and sudden jumps in the text: from the isoperimetric problem of the circle to measures of transcendency, or from fractional iterates to Soddy's formula for inscribed circles and Hilbert's fifth problem. For the same reason, light and loose reference is made to a few useful 'principles': telescoping cancellation, minimum perturbation principle, the principle of computing the same thing in two different ways, infinite crowding principle, symbol reification principle, etc."

However, just because an author decides on a plan of writing his book does not mean he is then unaccountable for the treatment even if it conforms 100% to his plan. An uncharitable person could very well say that parts of the prefaces were written after the books to justify the sketchy and uneven treatment of the many topics and problems. In my view, the reader would be much better served if there was not so much "fragmentation" in the treatment. I would have rather seen less material covered in a less sketchy fashion and a relegation of the material left out to formal exercises with adequate references. In regards to "exercises", there are quite a number of them, but they are not easy to find since they are distributed through the text. It would be more helpful to the reader if some of the problems were "set out" in the text and a clear indication given of their status, i.e., whether the problem is a known result, a conjecture or an open one. For example, on pp. 154–155(I), the author considers the problem of how many circles lying on a torus  $T$  can be drawn through an arbitrary point  $p$  of  $T$ . He then gives a nice proof that there are at least four such circles and invites the reader to show that there are exactly four. He ends with, "the reader may also wish to tackle the much harder problem of characterizing the torus by the four-circle property: (a) If  $S$  is a complete sufficiently smooth surface containing exactly four circles, through any point of it, then  $S$  is a torus; (b) If the number of such circles is  $> 5$ , then it is infinite and the surface is a sphere". Although the author gives no references, I was aware of the first problem from a problem in the 11th Putnam Competition [5] and from a paper on curves on a torus [6]. As to the last problems (a) and (b), I found them new and challenging and spent some time on them without success. Subsequently, on writing to the author for reprints of all his papers which are referred to, I also asked for references on the latter two problems. I also informed him I was in the process of reviewing his books. All I got back were some of the reprints that I requested. I wrote back another note specifically asking again about problems (a) and (b). This time I got back a note unbraiding me for communicating with the author since he considered that it would be unfair to the readers of my review. However, he was willing to answer questions if it came strictly (not related to the review) from one interested mathematician to another. And, on this basis, he admitted that (a) and (b) were conjectures. After replying to this note I refrained from any further communication even though I still have other similar questions to ask.

In regard to references, there are 79 in I and 188 in II. Unfortunately, at least to this reviewer, these are not nearly enough nor fully referenced to be adequate considering to whom the books are addressed and the too brief treatment of almost all the topics (one notable exception is the author's own approach to "Computing and Computability" in II). It is a disservice to the reader to find bald references, for example to the three volume treatise of Goursat-Hedrick [62] consisting of 1107 pages, or to the two volume treatise of Edwards [7] consisting of 1887 pages, without any specific page references. The references to journal articles are incomplete in that titles are not given. These would be most helpful in remembering and making cross-references and cross-connections of the texts' material. I also found the indexes not extensive enough. In this regard, I quote from R. Osserman's review of a book of Nitsche (Bull. Amer. Math. Soc., 82 (1976), 703–707) and which should be kept in mind by prospective authors:

"Let me digress a moment to wonder about the mentality of an author who takes months or years to write a book, and then begrudges the few days needed to compile a careful index that will enhance immensely the book's value to its readers. Nitsche not only makes every effort to direct an interested reader to the relevant parts of his book, but he does equal service in the opposite direction; when citing a result from another book or paper, he takes unusual pains to give precise page numbers—one more courtesy to his readers that this reviewer, for one, truly appreciates." "Every item listed has individual page references to each place that it is cited in the text, a simple device, but extraordinarily useful in determining the content (or at least the context of any of the 1232 papers and books listed in the bibliography). Since its usefulness far outweighs the effort needed in inserting these page references, I would hope that this practise might be widely adopted in future monographs." Amen!!

This practise is not that isolated, e.g., see Grunbaum [8] and Harary [9].

In II, there is not only a reference to Pólya, Szegő in the German edition but a reference to Vol. 1 in the new English edition which is commendable. However, the same should have been done with respect to Aczel's book in functional equations in German of 1961. Since there is also a revised English edition [10] with much new material and a bibliography that is almost twice the original size. In regard to "Iteration and Fractional Iteration" in I, I am surprised to find no reference to the book of Kuczma [11].

As for lapses of reference to specific problems, methods, tricks, etc., covered in I, II, one should not expect an author to reference everything since it is likely that some of these were developed independently by the author (and unpublished) or that they are so ingrained that the author has simply forgotten when, where and/or if he had obtained them from other sources. However, since I feel that there are too many such lapses and also since it may be helpful to prospective readers in regard to specific problems as well as in the general area of problem solving, I have indicated a number of these by volume and page in parentheses at the end of references [12] to [30]. Also, see [7-II, pp. 758, 768, 830], [16, pp. 46, 50, 237] for pp. 190–194 of vol. I; [7-I, p. 503] for p. 201 of vol. I; and [37, p. 392] for pp. 138–145 of vol. I.

On count there are at least 50 typographical errors. Fortunately, most of these are easily recognizable. There are also a number of misstatements which

are not too important (e.g., that  $\int_0^\infty (\log x)/(a+x)^2 dx$  cannot be evaluated by integration by parts; p. 197, I) and several other errors that are nontrivial. These are now listed together with some comments on several other problems.

I-2 (Page 2 of Vol. I). The author gives a sketchy proof of an interesting and a continually rediscovered result of Archimedes of an area preserving mapping of a sphere onto a closed right circular cylinder. Here the cylinder is circumscribed to the sphere and its height is congruent to a diameter. Points  $P$  on the sphere are mapped onto points  $P'$  on the cylinder by drawing a segment from  $P$ , perpendicular to the common axis of the two figures, intersecting the cylinder. Also, sketched, is Archimedes own proof. Then to have some application of this mapping result, he derives the formula for the area of a symmetric spherical quadrilateral (equal interior angles) using integration. Finally, using the spherical excess formula for a triangle, which is derived elegantly, he obtains the angles of the quadrilateral in terms of the sides. However, it is to be noted that:

(A) The elegant derivation of the area of a spherical triangle is due to Girard [32].

(B) The area for the quadrilateral of sides  $2\alpha$  and  $2\beta$  is derived as  $A = 4 \arcsin(\sin \alpha \sin \beta)$ . That this formula is incorrect follows by letting  $\alpha \rightarrow 0$  (then  $\beta \rightarrow \pi/2$ , producing a lune).

(C) The correct formula is  $A = 4 \arcsin(\tan \alpha \tan \beta)$  which can be obtained immediately from the known formula for a spherical quadrilateral inscribed in a small circle, i.e.,

$$\sin^2 A/4 = \frac{\sin(p-a)/2 \cdot \sin(p-b)/2 \cdot \sin(p-c)/2 \cdot \sin(p-d)/2}{\cos a/2 \cdot \cos b/2 \cdot \cos c/2 \cdot \cos d/2}$$

where  $p$  is the semiperimeter  $(a+b+c+d)/2$ .

(D) The formula in (B) would be correct if  $2\alpha$  and  $2\beta$  are taken as the angles of two symmetric lunes. A derivation is then elementary and is in reverse order. One first obtains the angles of the quadrilateral by finding the angle between two planes. Then one uses the spherical excess formula.

I-150. Here, the author gives a solution to the following problem: "A man in a boat, at a unit distance from a straight shore, finds himself lost in a completely impenetrable fog without knowing the direction of the shore. What is the shortest sailing curve the boat should follow, to make sure of hitting the shore?" The solution given here is incorrect. The problem had been treated previously by Isbell [32] who attributes the problem and related ones to Bellman. For other related search problems, see [33], [34].

I-224. For an application of the "Principle of Infinite Crowding" it is stated that "if infinitely many segments are subsets of  $[0, 1]$  and if the sum of their lengths is infinite, then some point of the interval  $[0, 1]$  is covered by infinitely many of the segments. A counterexample is given by  $I_n \equiv [1 - 2/n, 1 - 1/n]$ ,  $n = 2, 3, \dots$

II-70. The author gives some motivation leading to the inequality

$$\left\{ \sum_{i=0}^n a_i \right\}^2 > \sum_{i=0}^n a_i^3$$

where  $0 = a_0 < a_1 < \dots < a_n$  and  $a_{i+1} - a_i \leq 1$ ,  $i = 0, 1, \dots, n - 1$ , and then derives the integral counterpart

$$\left\{ \int_0^u f(x) dx \right\}^2 \geq \int_0^u f^3(x) dx, \quad u \geq 0, f(0) = 0, 0 < f'(x) \leq 1.$$

This is then reconsidered in reverse fashion on p. 393 where one of the conditions is replaced by the slightly different one:  $0 = a_0 \leq a_1 \leq a_2 \leq \dots$ . However, it is to be noted that the integral inequality was first set as a problem in the Putnam Competition [35] by D. J. Newman. Accompanying the problem was a hint. The author and J. S. Spouge, apparently piqued by the hint which was not a good one, wrote up a short note giving two proofs each of the integral inequality and its discrete analog and submitted it to the Amer. Math. Monthly. Coincidentally, I had been asked to referee the note and had immediately returned the note since I already had a joint note with D. J. Newman submitted to the Monthly with simple proofs of the two inequalities as well as extensions and probabilistic interpretations. I also communicated this dialog to the author and acknowledged his work in our paper [36]. Incidentally, when Newman proposed the integral inequality for the Putnam, he had forgotten about his simple proof and even the joint paper. The solution  $a_i = i$  given on p. 393 for the equality case is not quite correct. One can start off with an arbitrary number of zero terms.

II-301–316. This section consists of an interesting discussion on pursuit and related topics. In particular, consider a man travelling in a circle with unit speed and another man always heading directly towards him with speed  $v$ . It has been maintained by Hathaway and Davis that capture can only occur if  $v > 1$ . The author gives an intuitive argument to show that capture can occur if  $v < 1$ . Start both men together on the circle and have both move in the reverse sense (anti-pursuit). This leads to an initial position which if reversed again to direct pursuit will lead to capture. However, since the pursuit curve is singular at capture, one should justify the existence of a solution for the anti-pursuit curve. This can be done for the circle but does there exist an anti-pursuit curve for say the curve  $y = x^2 \sin 1/x$  starting at  $(0,0)$ ? Incidentally, the previous capture assertion had been made previously by Runkle and proved by Bernhart [38, p. 58]. It is of interest to note that Davis did not pick up this latter reference even though he had a reference to Bernhart's fourth paper on pursuit. This is understandable since Bernhart does not give explicit reference to his three previous pursuit papers.

II-327. To solve the functional equation  $F(x + y) + F(x - y) = 2F(x)F(y)$ , it is simpler to differentiate twice with respect to each variable and separate, i.e.,

$$F''(x + y) + F''(x - y) = 2F''(x)F(y) = 2F(x)F''(y),$$

$$F''(x)/F(x) = F''(y)/F(y) = \text{constant}.$$

Also, one can obtain the same solution by only assuming continuity of  $F$  (see [10]). To solve

$$F(x + y) = F(x)F(y)a^{xy}b^{xy(x+y)}$$

it is simpler to differentiate logarithmically and then separate.

In II-169 and the appendix, the author discusses a number of informal principles in problem solving. I would have preferred that he started his books with these principles and expanded upon them. This has been done essentially by Pólya in his five beautifully written books on problem solving [39]–[43]. Also, one may want to contrast the books here with Pólya-Szegő I, II [44], [45]. Even though the latter two books are classic and are much better organized, the Melzak books range wider and definitely have what A. N. Whitehead has called the “adventure of ideas”. Despite my previous criticisms, it is these “ideas” which should make the books appealing and worthwhile to many classes of readers, from well motivated high school students to professional mathematicians. However, to get at these “ideas”, the reader, in general, will have to work at them.

Since the author also mentions in the prefaces that the books are meant to accompany other texts, books, and instruction or self-instruction, I include some additional references for these [46]–[61]. This is particularly important with respect to modeling since the “concrete” descriptions given by the author are, in my view, much too brief for most classes of prospective readers.

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