describing an ensemble of k particles. Suppose that we are not allowed to observe trajectories directly, but only to observe the position of k particles at one fixed time. (Then we know that the predictions of the stochastic interpretation agree with the predictions of quantum mechanics.) We are free to impose any time-dependent potentials we wish and to consider $k - 1$ of the particles as observing instruments. How much information can we obtain about the trajectory of the remaining particle in this way?

The stochastic interpretation gives a clear meaning to the notion of the probability that a particle (in a process corresponding to a solution of the Schrödinger equation) is ever in a given region during a given interval of time. The orthodox theory of quantum mechanical measurement is restricted to observations made at one fixed time. Is there a quantum mechanical definition of this probability which agrees with the probability given by the stochastic interpretation?

There remains the problem of developing a stochastic relativistic theory. Theories of relativistic interaction appear to require fields. In recent years probabilistic techniques have played a large role in constructive quantum field theory, but the random fields have been constructed on Euclidean space, rather than Minkowski space, and the results for quantum fields have been obtained by analytic continuation. This is analogous to studying the Schrödinger equation by means of the corresponding heat equation, and then analytically continuing in time. The field-theoretic analogue of the stochastic interpretation of the Schrödinger equation remains to be constructed.

ADDED IN PROOF. Some of the questions raised here have been answered by David Shucker in a Princeton thesis (to appear).

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Grundzüge der universellen Algebra, by Herbert Lugowski, Teubner-Texte zur Mathematik, B. G. Teubner Verlagsgesellschaft, Leipzig, 1976, 238 pp., DM 19.50.

Universal algebra, as a method, has been extremely fruitful; by contrast, as an independent discipline it appears a little arid, owing to the fact that so many of its results have been somewhat less universal in their application. Perhaps the subject has developed best when working in harness with another part of mathematics, such as logic or category theory, and this is reflected in more recent books such as [1], [2]. Another field which would provide a good vantage point from which to view universal algebra is automata theory, and it cannot be long before the first book treating these two topics together will appear.

The book under review (printed from typescript) is on a rather more modest level; its aim is to make the reader familiar with the basic concepts and to illustrate them. One starting point for universal algebra consists in taking the three isomorphism theorems of E. Noether and restating them in a suitably general setting. Leaving operators aside for the moment, one thus arrives at the decomposition theorem for mappings: any mapping $f\colon X \to Y$ of sets can be decomposed into a surjection (of X on its quotient set defined by f), followed by a bijection, followed by an injection (the inclusion map of the image under f in Y). This gives rise to the factor theorem: given a mapping $f\colon X \to Y$ and any equivalence q on X which is mapped by f into the diagonal of Y^2, f can be uniquely factored by the natural mapping from X to its quotient set X/q. From this result it is easy to derive analogues of the isomorphism theorems for abstract sets; moreover when one comes to consider algebras, these results carry over to yield quite painlessly the usual isomorphism theorems. Here an 'algebra' is taken to mean a set A together with a set Ω of finitary operations on A (i.e. mappings $A^n \to A$, for varying n, the 'arity' of the operation). In practice one takes Ω to be an abstract set, graded so that with each $\omega \in \Omega$ an integer n, its arity, is associated; now an Ω-algebra structure is defined on a set A by prescribing for each n-ary operator ω in Ω an n-ary operation on A. This allows the notion of homomorphism to be defined (as a mapping compatible with the operators) and subalgebras, congruences (= equivalences admitting Ω) have the expected meaning. There are a number of formal results that can be proved at this stage, to wit the isomorphism theorems, characterizations of direct and subdirect products, which do not depend on the finitarity of the operations. Others, about generating sets, only involve finitarity in the most superficial way. It would have been interesting to see at this point the characterization of the lattice of subalgebras (as an algebraic closure system) but the author sticks to a more elementary approach. In this he is not always successful, for in his efforts to be intelligible he makes distinctions which at this level seem rather pedantic and then spends much of his time obliterating them later. Thus he defines free products in terms of presentations, and coproducts categorically (though categories get the barest mention) and then spends over 3 pages proving them equivalent.

A useful formal tool in the study of Ω-algebra is the Ω-word algebra or the absolutely free Ω-algebra. It is defined for a given generating set X by forming all possible combinations of X and Ω, e.g. if φ is binary and ω ternary, $xyz\varphi t\omega$ is possible, and it needs no brackets. The result is the algebra $W_\Omega(X)$ of Ω-words in X, which has every Ω-algebra generated by a set of the same cardinal as X as a homomorphic image. If we regard groups as algebras with one binary, one unary and one 0-ary operation (product, inverse, neutral element), the Ω-word algebra will not be a free group, because it does not satisfy the laws holding in groups (e.g. the associative law). This leads one to consider the class of Ω-algebras satisfying a given set of laws, a *variety* of Ω-algebras; e.g. groups form a variety in this sense. Each variety V has free algebras, algebras in V having all other members of V as homomorphic images. G. Birkhoff in 1933 proved that a class of Ω-algebras is a variety if and only if it is closed under subalgebras, direct products and homomorphic images, and this result is included here.

Not all the algebraic systems encountered in practice fall under the above notion of Ω-algebra. E.g. consider a ring R and an R-module M. For each $\alpha \in R$ we have an operation $x \to \alpha x$ and although it is possible to regard M as an algebra with a binary operation $+$ and a set R of unary operators, in many ways it is more natural to consider the pair M, R as the underlying carrier, e.g. given rings R, S, an R-module M and an S-module N we may wish to consider pairs of maps f: $R \to S$, g: $M \to N$, where f is a ring homomorphism and g an R-module homomorphism defining N as R-module by pullback along f: $n \cdot r = n(rf)$. To cope with this situation P. J. Higgins [3] introduced the notion of an algebra with a scheme of operators or Σ-algebra. Briefly, this is an algebra whose carrier is not a set, but an indexed family of sets, and with each n-ary operation the $n + 1$ sets housing the arguments and values are given. Besides modules over rings, one has the example of the set of all matrices over a ring R: if ${}^mR^n$ is the set of all $m \times n$ matrices, we have a Σ-algebra $\{{}^mR^n\}_{(m,n \in N)}$ with the usual operations. A further example is provided by automata; a Mealy machine is given by three sets X (input), Y (output), S (set of states) and two functions, the transition function δ: $S \times X \to S$ (giving the next state) and the output function λ: $S \times X \to Y$. But the allusion to automata is quite brief and is not followed up. For the next 30 pages the development runs almost entirely parallel to that of Ω-algebras and proofs are usually omitted. This section would have gained in interest if some applications to Σ-algebras had been mentioned (or even promised at a later stage).

A third chapter lists examples: groupoids, groups, rings, lattices and Boolean algebras. Exercises that should be left to the reader (How can a congruence on a group be represented by a normal subgroup?) are proved in some detail; some less obvious facts are also proved, e.g. it is shown that epimorphisms of groups are surjective, but of rings are not. A six page appendix lists further results and open problems.

Clearly this is a book aimed at beginners, but in a first exposition great stress needs to be laid on the intuitive content of the results. If this had been done the book could have been made much more readable and its value as a text-book enhanced.

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