

## ON LIE ALGEBRAS OF DIFFERENTIAL FORMAL GROUPS OF RITT

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Communicated by Hyman Bass, May 20, 1977

Between February 1949 and August 1950 J. F. Ritt published four papers [8]–[11] about differential formal groups. This research was stopped by his death on January 5, 1951. Below we describe some results which can be considered as a continuation of the research begun in the abovementioned papers of Ritt.

1. Let  $K$  be an algebraically closed field of characteristic 0 and let  $k$  be a subfield of  $K$ . A finite-dimensional vector  $K$ -space  $D$  is called (after W. Y. Sit) a Lie  $K$ -space if  $D$  has a structure of a Lie  $k$ -algebra. Suppose now that  $D \subseteq \text{Der}_k K$  and that  $k = \{a \in K \mid Da = 0\}$ . We denote by  $K[D]$  the (associative) algebra of differential operators; it is generated by  $K$  and  $D$  with relations  $d\lambda = \lambda d + d(\lambda)$ ,  $d_1 d_2 - d_2 d_1 = [d_1, d_2]$  for  $d, d_1, d_2 \in D$ ,  $\lambda \in K$ . If  $M, N$  are  $K[D]$ -modules then  $M \otimes_K N$  and  $\text{Hom}_K(M, N)$  are given natural structures of  $K[D]$ -modules by  $d(m \otimes n) = dm \otimes n + m \otimes dn$  and  $(d\varphi)(m) = d(\varphi(m)) + \varphi(-dm)$  for  $\varphi \in \text{Hom}_K(M, N)$ ,  $m \in M$ ,  $n \in N$ ,  $d \in D$ .

1.1 DEFINITION. A  $K[D]$ -algebra  $A$  is a map  $A \otimes_K A \rightarrow A$  of  $K[D]$ -modules. A  $K[D]$ -coalgebra  $C$  is a map  $C \rightarrow C \otimes_K C$  of  $K[D]$ -modules. A  $K[D]$ -bialgebra  $B$  is a  $K[D]$ -algebra together with a  $K[D]$ -coalgebra structure  $B \rightarrow B \otimes_K B$  which is a map of  $K[D]$ -algebras.

1.2 DEFINITION. A  $K[D]$ -module  $M$  is split if  $M = K \otimes_k M^0$  for some  $k$ -module  $M^0$  and  $DM^0 = 0$ . The split action of  $d \in D$  will be denoted  $d^0$ .

1.3 REMARK. If  $M$  is a split  $K[D]$ -module and we have another action of the elements  $d \in D$  on  $M$  which makes  $M$  into a  $K[D]$ -module, then  $d - d^0 \in \text{End}_K M$ . For  $\omega(d) = d - d^0$  we have the relation

$$\omega([d_1, d_2]) = d_1^0 \omega(d_2) - d_2^0 \omega(d_1) + [\omega(d_1), \omega(d_2)].$$

Therefore  $\omega(d)$  can be considered as a differential 1-form with values in the Lie  $K$ -space  $\text{End}_k M$ , and the above relation reduces to  $\delta\omega = -\frac{1}{2}[\omega, \omega]$ , where  $\delta$  denotes the exterior differentiation of differential forms, given by the formula

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AMS (MOS) subject classifications (1970). Primary 14L05, 14L25, 17B65, 22E65, 54B15.

<sup>1</sup>Supported by the National Science Foundation.

$$(\delta\omega)(d_1, d_2) = \frac{1}{2}\omega([d_1, d_2]) - \frac{1}{2}d_1^0\omega(d_2) + \frac{1}{2}d_2^0\omega(d_1).$$

If  $M$  has a split  $K[D]$ -algebra structure and we have another action of the elements  $d \in D$  on  $M$  which makes the  $K$ -algebra  $M$  into a  $K[D]$ -algebra then  $\omega(d) \in \text{Der}_K M$ .

1.4 The group  $\text{Aut}_K M$  of  $K$ -module or  $K$ -algebra automorphisms of a split  $K[D]$ -structure on  $M$  acts on the set of all  $K[D]$ -structures by  $(g^{-1}\omega)(d) = g^{-1} \cdot d(g) + g^{-1}\omega(d)g$ . Here we denote by  $d(g)$  the action of  $d \in D$  on  $g \in \text{End}_K M = \text{Hom}_K(M, M)$ . Denoting by  $g^{-1} \cdot \delta g$  the differential 1-form  $d \rightarrow g^{-1} \cdot d(g)$  we have  $g^{-1}(\omega) = g^{-1} \cdot \delta g + g^{-1}\omega g$ , i.e., the equivalence class of  $\omega$  is a connection ("in a principal bundle with fiber  $\text{End}_K M$ ").

1.5 If the  $K[D]$ -module  $M$  is finitely generated then the  $K[D]$ -module  $M^*$  has the linearly compact topology [4] and  $M = (M^*)^*$  where the second star denotes the topological dual.

**2. Definition.** A regular complete local commutative  $K[D]$ -finitely generated continuous  $K[D]$ -bialgebra  $A$  is called a differential formal group. (Ritt considered the case in which  $\dim D = 1$  and  $A$  is the algebra of formal power series over a  $K[D]$ -free module.)

Let  $J$  be the maximal ideal of  $A$  (where  $A$  is as above). Then the vector space  $(J/J^2)^*$  is given, as usual, the structure of a Lie algebra, called the Lie algebra of the formal group. In our case the Lie algebra will be a linearly compact  $K[D]$ -algebra.

3. Our aim is to state some structure results about linearly compact Lie  $K[D]$ -algebras  $G$  such that  $G^*$  (topological dual) is finitely generated. We denote the set of such algebras by  $\mathfrak{G} = \mathfrak{G}(D)$ .

3.1 PROPOSITION. *Let  $G \in \mathfrak{G}$ . Then  $G$  has a greatest solvable  $K[D]$ -ideal. It is the sum of all solvable  $K[D]$ -ideals and its derived series terminates at zero in a finite number of steps.*

3.2 PROPOSITION. *Let  $G \in \mathfrak{G}$  be a  $K[D]$ -simple algebra. Then at least one of the following holds,*

(i)  *$G$  is  $K$ -simple and therefore (cf. [5]) either finite dimensional or of Cartan type  $W_n, S_n, H_n, K_n$ .*

(ii) *There exists a Lie  $K$ -subspace  $\tilde{D}$  and a  $K$ -simple Lie  $K[\tilde{D}]$ -algebra  $S \in \mathfrak{G}(\tilde{D})$  such that  $G = \text{Hom}_{K[\tilde{D}]}(K[D], S)$ .*

The proof of (ii) in the above proposition is modelled on Blattner's proof [1], [2] of a Guillemin result [4]. The difficulty is that  $D$  acts on  $K$  and that  $K[D]$  is not a bialgebra.

4. The above results show that completely to describe the simple  $K[D]$ -algebras  $\mathfrak{G}(D)$  it is sufficient to consider only  $K$ -simple ones. To begin with we

note that such algebras always have a split  $K[D]$ -structure (because  $K$  is algebraically closed). As usual we denote this structure by  $d^0$ . If  $G$  is not finite dimensional and  $G$  has the split structure then  $G^*$  is not finitely generated. Let  $\omega(d) = d - d^0$  as above. First we have

**THEOREM.** *If  $G \in \mathfrak{G}(D)$  is simple and finite dimensional then for every differential 1-form  $\omega$  satisfying  $\delta\omega = -\frac{1}{2}[\omega, \omega]$ , there exist an extension  $L$  of  $K$  and  $g \in (\text{Aut } G)(L)$  such that  $\omega$  is an exact differential:  $\omega = g^{-1} \cdot \delta g$ .*

The proof is modelled on proofs of Cassidy [3] and Kovacic [7] and also was suggested by J. Tits.

5. In the case when  $G$  is of Cartan type, I was able to obtain conclusive results only in the ‘‘Euclidean’’ case, i.e. the case in which  $D$  has a commutative basis. (J. Hrabowski has informed me recently that this condition holds automatically.) We recall that  $G$  is graded  $G = \sum_{i \geq -2} G_i$  and therefore  $\text{Der}_K G = \sum_{i \geq -2} (\text{Der}_K G)_i$ . Write  $\omega(d) = \sum \omega(d)_i$ ,  $\omega(d)_i \in (\text{Der}_K G)_i$ . We define  $\psi_\omega : D \rightarrow \sum_{i < 0} (\text{Der}_K G)_i$  by  $\psi_\omega(d) = \sum_{i < 0} \omega(d)_i$ .

5.1 **THEOREM.** *Suppose that  $D$  has a commutative basis and  $G \in \mathfrak{G}$ . Then  $\psi_\omega(D) = \sum_{i < 0} (\text{Der}_K G)_i$ .*

Let  $\tilde{D} = \text{Ker } \psi_\omega$ . Then  $\tilde{D}$  is a Lie  $K$ -subspace of  $D$  of codimension  $n = \sum_{i < 0} \dim(\text{Der}_K G)_i$ . The set of all such subspaces in  $D$  we denote  $\Gamma_n$ . Let  $\Pi(G)$  denote the set of  $K[D]$ -structures on  $G$  which belong to  $\mathfrak{G}$ . The above theorem defines a map  $\pi : \Pi(G) \rightarrow \Gamma_n$ .

5.2 **THEOREM.** (i) *If  $G$  is of type  $W_n$  then  $\pi$  is a bijection of  $\Pi(G)$  and  $\Gamma_n$ .*

(ii) *If  $G$  is of type  $S_n$  or  $H_n$  then the fibers of  $\pi$  are subsets of  $K^*/k^*$  or  $\text{GL}(n, K)/\text{Sp}(n, K)k^*$  respectively.*

(iii) *If  $G$  is of type  $K_n$  then the fibers of  $\pi$  are subsets of some algebraic homogeneous space for  $\text{GL}(n, K)$ .*

6. The above suggests the following definition of an affine  $K[D]$ -algebraic variety: it is the prime spectrum of a commutative  $K[D]$ -finitely generated  $K[D]$ -algebra. In particular, an affine  $K[D]$ -algebraic group is  $\text{Spec } A$  where  $A$  is a commutative  $K[D]$ -bialgebra. In particular, this group (considered as a  $K$ -group) is a projective limit of finite dimensional affine algebraic  $K$ -groups. This shows that the Lie algebras of Cartan type are not Lie algebras of affine  $K[D]$ -algebraic groups (if the above definition is accepted). Our definition differs from that of Cassidy [3].

7. The algebras studied by A. A. Kirillov in [6] are (modulo the replacement of a ring by a field and  $C^\infty$  by analyticity) our algebras with the additional condition that  $G^*$  have one generator. Kirillov’s condition of transitivity im-

plies that they are of Cartan type, so they are either  $K_n$  or the one-dimensional central extension of  $H_n$ .

8. To conclude this announcement I express my deep gratitude to V. Guillemin, D. Kazhdan and E. Kolchin who had patience to listen to me and who gave me much help.

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