

## AVERAGE GAUSSIAN CURVATURE OF LEAVES OF FOLIATIONS

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Let  $F$  be a smooth transversely-oriented foliation of a compact, connected, oriented, Riemannian manifold  $W^{n+1}$  of constant sectional curvature  $\equiv c$ . Let  $K_F: W \rightarrow \mathbf{R}$  via  $K_F(x) =$  the Gaussian curvature (defined below) of the leaf  $l^n$  through  $x$  at  $x$ . For  $n = 2$  this is classical Gaussian curvature. Let  $\text{vol}$  be the canonical volume on  $W$ , and define  $\bar{K}_F$  by  $\text{Volume}(W) \cdot \bar{K}_F = \int_W K_F \text{vol}$ .

THEOREM 1.

$$\bar{K}_F = \begin{cases} 2^n c^{n/2} / \binom{n}{n/2}, & n \text{ even,} \\ 0, & n \text{ odd.} \end{cases}$$

THEOREM 2. Let  $n + 1 = 3$  and suppose  $F, W, c$  are as above except that  $\partial W$  is nonempty and is a union of leaves of  $F$ . Then

$$\int_W K_F \text{vol} = 2c \text{Volume}(W) + \int_{\partial W} H \text{vol}'$$

where  $H: \partial W \rightarrow \mathbf{R}$  is the mean curvature (computed with respect to the transverse orientation), and  $\text{vol}'$  is the canonical volume on  $\partial W$ .

THEOREM 3. Suppose  $n + 1 = 3$ . Let  $F$  and  $W$  be as in the original hypotheses with  $\partial W = \emptyset$  but assume the sectional curvatures of  $W$  lie between  $c_1$  and  $c_2$ . Then we have  $2c_1 \leq \bar{K}_F \leq 2c_2$ .

DEFINITION OF GAUSSIAN CURVATURE. We define, for a Riemannian manifold  $l = l^n$ , the function  $K: l \rightarrow \mathbf{R}$  in two cases (which overlap):

Case (i).  $n$  is even. In this case a local orthonormal frame on  $l$  gives rise to a matrix of curvature 2-forms,  $\Omega = (\Omega_j^i)$  defined locally. The Pfaffians of the local  $\Omega$  agree on overlaps and so define a global  $n$ -form  $\text{Pf}(\Omega)$  on  $l$ . Letting  $\nu$  denote the canonical volume form on  $l$  we set

$$K\nu = \frac{2^{n/2} \cdot (n/2)!}{n!} \text{Pf}(\Omega)$$

(see [3, vol. V, pp. 417–420]).

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Case (ii). Assume  $l$  is a hypersurface of a flat Riemannian manifold  $W$ , and that  $l$  is transversely oriented by a field of unit normals  $\xi$ . Then at each point  $x$  of  $l$  let  $A_x: T_x l \rightarrow T_x l$  be defined by  $A_x v = -\nabla_v \xi$ . Then we define  $K(x) = \det(A_x)$ . (See [3, vol. IV, p. 96].)

REMARKS. In the overlap of Cases (i) and (ii), viz. when  $l$  is an even-dimensional hypersurface of a flat manifold, the two definitions of  $K$  agree. If  $n$  is even then  $K$  is intrinsic to the geometry of  $l$ ; if  $n \geq 3$  is odd then  $K$  is intrinsic up to a global choice of sign [3, vol. IV, p. 96].

SKETCH OF PROOF OF THEOREM 1. We consider two cases:  $n$  odd and  $n$  even.

(i) The case  $n$  is odd:

Here  $\chi(W) = 0$  and hence by Chern-Gauss-Bonnet [3, vol. V, p. 442] the constant curvature  $c = 0$ , i.e.  $W$  is flat. Without loss of generality we may assume, by taking a finite covering, that  $W$  is in fact a flat torus [1, p. 212].

Let  $T_p \approx \mathbb{R}^{n+1}$  denote the tangent space to  $W$  at some point  $p \in W$ . A choice of unit normal vector field  $\xi$  to the foliation  $F$  determines (by parallel translation in  $W$ ) a Gauss map  $g: W \rightarrow T_p$  whose image lies of course in the unit sphere  $S^n \subset T_p$ . Think of  $dg$  as a map  $dg: W \rightarrow \text{End}(TW)$  via  $x \mapsto dg_x$ .

Let  $\sigma_i(E_x)$  denote the  $i$ th elementary symmetric function of the eigenvalues of  $E_x$ , where  $E_x$  is any endomorphism  $E_x: T_x \rightarrow T_x$ .

LEMMA.  $K_f(x) = \sigma_n(-dg_x)$ , for all  $x \in W$ .

The proof is not difficult.

Now for each  $t \in \mathbb{R}$  consider  $h_t: W \rightarrow W$  defined by  $h_t(x) = \exp(tg(x))$ , or in other words  $h_t(x) = x + tg(x)$  (by slight abuse of notation). A computation shows that

$$\int_W Jh_t \text{vol} = \int_W \det(I + tdg) \text{vol} \quad \text{or}$$

$$(*) \quad \text{Volume}(W) = \text{Volume}(W) \cdot [1 + \bar{\sigma}_1(dg)t + \dots + \bar{\sigma}_n(dg)t^n]$$

where  $\bar{\sigma}_i(dg)$  denotes the average over  $x \in W$  of  $\sigma_i(dg_x)$ , and  $J$  denotes the Jacobian.

Since both sides of (\*) are polynomials in  $t$  it follows that  $\bar{\sigma}_i(dg) = 0, i = 1, \dots, n$ .

COROLLARY. In the above case we have  $\bar{\sigma}_i(dg) = 0$  for  $i = 1, \dots, n$ . In particular  $\sigma_2(dg)$  is a multiple of the leaf scalar curvature; hence the average leaf scalar curvature is 0 whenever  $W$  is flat.

SKETCH OF PROOF OF THEOREM 1 (CONTINUED).

(ii) The case  $n$  is even:

The proof depends on the construction of certain globally defined  $n$ -forms.

Let  $\{\theta^1, \dots, \theta^n, \theta^{n+1}\}$  be a local adapted orthonormal coframe field (with

$\theta^{n+1}$  orthogonal to the leaves of  $F$ ) and let  $\{\omega_j^i\}$  be the associated Riemannian connection forms. Put

$$\phi_r = \sum_{\sigma \in S_n} (-1)^\sigma \omega_{n+1}^{\sigma(1)} \wedge \dots \wedge \omega_{n+1}^{\sigma(2r-1)} \wedge \theta^{\sigma(2r)} \wedge \dots \wedge \theta^{\sigma(n)}$$

for  $1 \leq r \leq n/2$ , where  $S_n$  denotes the symmetric group on  $\{1, \dots, n\}$  and  $(-1)^\sigma$  is the sign of the permutation  $\sigma$ .

LEMMA. *The  $n$ -forms  $\phi_r$  do not depend on the choice of orthonormal coframe  $\{\theta^i\}$  and hence are globally defined on  $W$ .*

The proof is an unpleasant calculation.

LEMMA. *For each  $n$  there exist constants  $b_r$ ,  $1 \leq r \leq n/2$  such that if we set*

$$\Phi = \sum_{r=1}^{n/2} b_r \phi_r \quad \text{then}$$

(\*\*)

$$d\Phi = (K_F - a_n c^{n/2}) \text{vol} \quad \text{where } a_n = 2^n / \binom{n}{n/2}.$$

The proof is an even more unpleasant calculation.

Integrating (\*\*) over  $W$  readily yields  $\bar{K}_F = 2^n c^{n/2} / \binom{n}{n/2}$  as desired.

REMARKS. By taking double covers we may prove Theorem 1 even if  $W$  is allowed to be nonorientable. If  $n$  is even then we may similarly drop the assumption that  $F$  is transversely orientable. If  $n$  is odd, however, transverse orientability is required in order that  $K_F$  be defined.

Theorem 1 has been generalized in various ways in the recent paper of Rosenberg, Brito and Langevin [2]. Theorems 2 and 3 are proved using methods similar to Theorem 1.

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