BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 1, January 1978

NATURAL STRUCTURES ON SEMIGROUPS WITH INVOLUTION

BY MICHAEL P. DRAZIN

Communicated by Olga Taussky Todd, January 21, 1977

Let S be any semigroup, i.e. a set furnished with an associative binary operation, denoted by juxtaposition. An *involution* on S will mean a bijection $u \longrightarrow u^*$ of S onto itself, satisfying $(a^*)^* = a$, $(ab)^* = b^*a^*$. Such an involution is called *proper* (cf. [3, p. 74]) iff $a^*a = a^*b = b^*a = b^*b$ implies $a = b$. A *proper $*$-semigroup* will mean a pair $(S, *)$ where S is a semigroup and $*$ is a specified proper involution of S; in practice, we write S as an abbreviation for $(S, *)$. From now on, S will denote an arbitrary proper $*$-semigroup.

Obvious natural special cases are

(i) all proper $*$-rings, with "properness" (Herstein [4, p. 794] prefers to say "positive definiteness") as customarily defined (cf. [5, p. 31], [1, p. 10]) via $u^*u = 0$ implying $u = 0$ (in particular, with the obvious choices for $*$, all commutative rings with no nonzero nilpotent elements, all Boolean rings, the ring $\mathcal{B}(H)$ of all bounded linear operators on any complex Hilbert space H, and the ring $M_n(\mathbf{C})$ of all $n \times n$ complex matrices), and, only slightly less trivially,

(ii) all inverse semigroups (in particular, all groups).

We make a start here towards showing that the proper $*$-semigroup axioms allow the simultaneous development of a surprisingly rich common theory of these special cases. While there is clearly little likelihood of learning anything new about groups or Boolean rings by such an approach, none of the results about proper $*$-semigroups which we state below has previously been noted even for $n \times n$ matrices (still less for $\mathcal{B}(H)$ or for $*$-rings), and most provide new information also about inverse semigroups.

We recall that an element $a \in S$ is called *regular* iff $a \in aSa$, and *$*$-regular* iff there is an $x \in S$ with

$$axa = a, \quad xax = x, \quad (ax)^* = ax, \quad (xa)^* = xa.$$

For given $a \in S$, there can (cf. [6, Theorem 1]) be at most one such x, and, if any x exists, we write $x = a^\dagger$ (known as the *Moore-Penrose generalized inverse* of a). It is known (cf. [7]) that a is $*$-regular iff aa^* and a^*a are both regular; let $V_*(S)$ denote the set of all such a. (For example, $V_*(\mathcal{B}(H))$ consists [3, p. 73]

AMS (MOS) subject classifications (1970). Primary 06A10, 15A09, 16A28, 20M99, 46K99; Secondary 16A30.

of all operators with closed range, while $V_*(S) = S$ whenever every element of S is regular, e.g. if S is $M_n(\mathbf{C})$ or any inverse semigroup.)

We introduce a binary relation

$$(\Omega^*) \qquad a^*a = a^*b = b^*a \quad \text{and} \quad aa^* = ab^* = ba^*$$

on S, and another on $V_*(S)$:

$$(\Omega^\dagger) \qquad a^\dagger a = a^\dagger b = b^\dagger a \quad \text{and} \quad aa^\dagger = ab^\dagger = ba^\dagger .$$

THEOREM 1. *Ω^* is a partial order on the set S, and coincides with $\Omega^\dagger$ on $V_*(S)$.*

Thus each of Ω^*, $\Omega^\dagger$ is a natural generalization both of the usual partial ordering of the projection elements in a proper $*$-ring (in particular, the set of all elements of any Boolean ring) and also of Vagner's natural partial order (see e.g. [2, p. 40]) defined on the elements of any inverse semigroup. We write $a \leqslant b$ when Ω^* holds.

COROLLARY 1. *If $a, b \in V_*(S)$, then $a \leqslant b$ iff $a^* \leqslant b^*$ iff $a^\dagger \leqslant b^\dagger$.*

THEOREM 2. *For any $a \in V_*(S)$, we have*

$$\begin{aligned} a^\dagger &= \max\{y : y \in V_*(S), yay = y, (ay)^* = ay, (ya)^* = ya\} \\ &= \min\{y : y \in V_*(S), aya = a, (ay)^* = ay, (ya)^* = ya\}, \end{aligned}$$

where the max *and* min *are with respect to the partial order Ω^*.*

THEOREM 3. *If a given element b of S is either regular, $*$-regular, idempotent, a projection or a partial isometry, then every $a \in S$ satisfying $a \leqslant b$ inherits the same property.*

THEOREM 4. *If $a, d \in S$ and $b, c \in V_*(S)$, then*

$$\text{(i)} \qquad ab = cd \text{ and } a^*b = cd^*$$

holds iff

$$\text{(ii)} \qquad c^\dagger a = db^\dagger \text{ and } c^\dagger a^* = d^*b^\dagger .$$

Theorem 4 generalizes a result of Foulis [3, p. 83], who considered only the case $a = d$, $b = c$.

Further types of structure are revealed by considering Ω^*, $\Omega^\dagger$ and the Moore-Penrose map $a \longrightarrow a^\dagger$ in combination with certain other binary relations on S:

DEFINITION 1. Write $a \approxeq b$ iff either $a = b$ or there exist $p, q \in S$ with $a = pbq$, $qap = b$, $ap = pb$ and $qa = bq$.

DEFINITION 1*. Write $a \overset{*}{\approxeq} b$ iff either $a = b$ or there exists $p \in S$ with $a = pbp^*$, $p^*ap = b$, $ap = pb$ and $p^*a = bp^*$.

DEFINITION 2. Write $a \sim b$ iff there exist $y, z \in S$ with $a = aybza$, $b = bzayb$.

DEFINITION 2*. Write $a \overset{*}{\sim} b$ iff there exist $u \in aSb$ and $v \in bSa$ such that $aa^* = uu^*$, $b^*b = u^*u$, $a^*a = v^*v$, $bb^* = vv^*$.

DEFINITION 3. Write $a \approx b$ iff $a \in Sbs$ and $b \in SaS$.

DEFINITION 3*. Write $a \overset{*}{\approx} b$ iff there exist $u, w \in aSb$ and $v, x \in bSa$ such that $aa^* = uu^*$, $b^*b = w^*w$, $a^*a = v^*v$, $bb^* = xx^*$.

Of course $\sim$ and $\overset{*}{\sim}$ are natural extensions, to $*$-semigroups, of equivalences which are standard (for idempotents and projections respectively) in $*$-ring theory (see e.g. [1], [5]).

THEOREM 5. *Each of* $\approxeq$, $\overset{*}{\approxeq}$, $\sim$, $\overset{*}{\sim}$, $\approx$, $\overset{*}{\approx}$ *is an equivalence relation when restricted to the set of all regular elements of S.*

Let $\simeq$ and $\overset{*}{\simeq}$ denote ordinary similarity and unitary similarity.

THEOREM 6. *On* $V_*(S)$, *we have the implications*

$$\begin{array}{ccccccc} \overset{*}{\simeq} & \Longrightarrow & \overset{*}{\approxeq} & \Longrightarrow & \overset{*}{\sim} & \Longrightarrow & \overset{*}{\approx} \\ \Big\Downarrow & & \Big\Downarrow & & \Big\Downarrow & & \Big\Downarrow \\ \simeq & \Longrightarrow & \approxeq & \Longrightarrow & \sim & \Longrightarrow & \approx \end{array}$$

(*where each of the implications involving* $\overset{*}{\simeq}$ *or* $\simeq$ *applies only in the presence of a unity element*).

THEOREM 7. *If* $a \in V_*(S)$, *then* $a \sim a^*$ *and* $a \sim a^\dagger$.

COROLLARY 2. *If* $a, b \in V_*(S)$, *then* $a \sim b$ *iff* $a^* \sim b^*$ *iff* $a^\dagger \sim b^\dagger$ (*and similarly for* $\approx$).

THEOREM 8. *On* $V_*(S)$, *each of* $\approx$, $\overset{*}{\approx}$ *satisfies the Schröder-Bernstein property with respect to* Ω^*.

Proofs, applications and other related results will appear elsewhere.

REFERENCES

1. S. K. Berberian, *Baer $*$-rings*, Springer-Verlag, Berlin and New York, 1972.

2. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, vol. 2, Math. Surveys, no. 7, Amer. Math. Soc., Providence, R.I., 1967.

3. D. J. Foulis, *Relative inverses in Baer $*$-semigroups*, Michigan Math. J. 10 (1963), 65–84.

4. I. N. Herstein, *On rings with involution*, Canad. J. Math. 26 (1974), 794–799.

5. I. Kaplansky, *Rings of operators*, Benjamin, New York, 1968.

6. R. Penrose, *A generalized inverse for matrices*, Proc. Cambridge Philos. Soc. 51 (1955), 406–413.

7. N. S. Urquhart, *Computation of generalized inverse matrices which satisfy specified conditions*, SIAM Review 10 (1968), 216–218.

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907