

REPRESENTATION THEOREMS FOR MULTIFUNCTIONS AND ANALYTIC SETS

BY ALEXANDER D. IOFFE

Communicated by Hans Weinberger, May 9, 1977

We denote by (T, Σ) a measurable space and by X, Y, Z metrizable topological spaces. A *multifunction* from T into X is a mapping $t \rightarrow M(t)$ which assigns a set $M(t) \subset X$ (possibly empty) to each $t \in T$. The set

$$\text{dom } M = \{t \in T \mid M(t) \neq \emptyset\}$$

is called the *domain* of M . The multifunction M is called Σ -*measurable* if

$$M^{-1}(C) = \{t \in T \mid M(t) \cap C \neq \emptyset\}$$

belongs to Σ whenever $C \subset X$ is closed.

If $(T, \Sigma), (T', \Sigma')$ are measurable spaces, then $\Sigma \otimes \Sigma'$ denotes the product σ -algebra generated by rectangles $E \times E', E \in \Sigma, E' \in \Sigma'$. By $B(X)$ we denote the algebra of Borel subsets of X . A set $E \subset T$ is called Σ -*analytic* if it can be represented as a result of the A -operation of Suslin applied to elements of Σ . An equivalent definition: E is Σ -analytic if there is a Polish space X and a set $A \in \Sigma \otimes B(X)$ such that E is the projection of A on T (cf. [3]).

THEOREM 1. *Let X be a Polish (resp. compact metrizable) space, and let M be a Σ -measurable multifunction from T into X such that the sets $M(t)$ are closed. Then there exist a Polish (resp. compact metrizable) space Z and a mapping $f(t, z): T \times Z \rightarrow X$ such that*

- (i) f is continuous in z and Σ -measurable in t ;
- (ii) for all $t \in \text{dom } M$, one has $M(t) = f(t, Z)$, the range of $f(t, \cdot)$.

If X is a separable Fréchet space and all sets $M(t)$ are convex, then there is a pair (Z, f) satisfying (i), (ii) and such that Z is a closed convex subset in another separable Fréchet space Y , and for all $t \in T$, the mapping $z \rightarrow f(t, z)$ is the restriction on Z of a linear nonexpansive mapping from Y into X .

Taking a dense countable set $\{z_1, \dots\}$ in Z , one gets a dense countable family of measurable selectors for M . The existence of such a family was established by Castaing [2]. In our case, however, this family is rather "well arranged". In fact, the method we have used to prove the theorem demands explicit construction of such a family (with the help of the selection theorem of Rokhlin [8] and

AMS (MOS) subject classifications (1970). Primary 28A40; Secondary 49E10.

Kuratowski and Ryll-Nardzewski [6]), to define the space Z and the mapping f . The latter is carried out much in the same manner as in Ekeland's work [4], in which a similar result for continuous convex-and-compact-valued multifunctions was proved.

We shall say that the measurable space is *complete* if Σ contains all Σ -analytic subsets of T .

THEOREM 2. *Assume that (T, Σ) is a complete measurable space and that X is a Lusin space. Let $A \subset T \times X$ be $\Sigma \otimes B(X)$ -analytic, and denote by E the projection of A on T . Then there are a Polish space Z and a mapping $f: E \times Z \rightarrow X$ which is Σ -measurable in t , continuous in z and such that A is the range of the mapping $(t, z) \rightarrow (t, f(t, z))$ from $E \times Z$ into $E \times X$.*

V. Levin has brought my attention to the fact that this theorem can be restated in a stronger form: it is possible to assume the measurable space complete but require instead that f be measurable with respect to the σ -algebra generated by Σ -analytic subsets of T , rather than with respect to Σ .

For any $z \in Z$, the mapping $t \rightarrow f(t, z)$ is a measurable selector for A . Thus Theorem 2 implies a result on selectors which turns out more general than the corresponding results of Aumann [1] and Sainte-Beuve [9], because our assumptions on the measurable space are, in fact, weaker. (There is a complete measurable space whose algebra does not contain all universally measurable sets.) Note that selection theorems of such type (for analytic sets) go back to Lusin [7] and Yankov [10].

THEOREM 3. *Let X be a compact metrizable space, and S be a Banach space of continuous mappings from X into R^n such that the imbedding $i: S \rightarrow C(X, R^n)$ is continuous. Assume also that either $i(S)$ is F_σ in $C(X, R^n)$ or the measurable space (T, Σ) is complete.*

Let there be given a multifunction $(t, x) \rightarrow Q(t, x)$ from $T \times X$ into R^n such that

- (a) $Q(t, x)$ is nonempty convex and compact for each t, x ;
- (b) for all $x \in X$, the multifunction $t \rightarrow Q(t, x)$ is measurable; for all $t \in T$, the multifunction $x \rightarrow Q(t, x)$ is Hausdorff continuous;
- (c) for all t, x and all $z \in Q(t, x)$, there is a mapping $h(\cdot) \in S$ such that $h(x) = z$ and $h(u) \in Q(t, u)$ for all $u \in X$.

Then there are a Polish space V and a mapping $g: T \times X \times V \rightarrow R^n$ such that

- (i) g is measurable in t and continuous in (x, v) ;
- (ii) $g(t, \cdot, v) \in S$ for any t, v ;
- (iii) $Q(t, x) = g(t, x, V)$ for all t, x .

It follows from the theorem that the differential inclusion $\dot{x} \in Q(t, x)$ can be rewritten as an ordinary differential equation with control $\dot{x} = g(t, x, v)$ in

such a way that g will preserve regularity properties of Q . For instance g can be taken Lipschitz in x if $Q(t, \cdot)$ is Lipschitz, or C^1 in x if for every t the multifunction $Q(t, \cdot)$ admits a rich collection of C^1 -selectors. The fact that such reduction is possible with g merely continuous in x was proved earlier by Ekeland and Valadier [5].

REFERENCES

1. R. Aumann, *Measurable utility and the measurable choice theorems*, Proc. Coll. Int. CNRS, Aix-en-Provence, 1967, Editions du Centre Nat. Recherche Sci., Paris, 1969.
2. Ch. Castaing, *Sur les multi-applications mesurables*, Rev. Française Informat. Recherche Opérationnelle 1 (1967), 91–126.
3. C. Dellacherie et P.-A. Meyer, *Probabilités et potentiel*, Hermann, Paris, 1975.
4. I. Ekeland, *Paramétrisation en théorie de la commande*, C. R. Acad. Sci. Paris 267 (1968), 98–100.
5. I. Ekeland and M. Valadier, *Representations of set-valued mappings*, J. Functional Analysis (to appear).
6. K. Kuratowski and C. Ryll-Nardzewski, *A general theorem on selectors*, Bull. Acad. Polon. Sci. Sér. Math. Astronom. Phys. 13 (1965), 397–411.
7. N. N. Lusin, *Sur la problème de J. Hadamard d'uniformisation des ensembles*, Mathematica 4 (1930), 54–66.
8. V. A. Rokhlin, *Selected topics from the theory of dynamical systems*, Amer. Math. Soc. Trans. (2) 49 (1966), 171–240.
9. M. F. Sainte-Beuve, *Sur la généralisation d'un théorème de section de von-Neumann-Aumann*, Travaux Sem. d'Analyse Convexe, vol. 3, Univ. du Languedoc, 1973.
10. V. Yankov, *On the uniformization of A-sets*, Dokl. Akad. Nauk SSSR 30 (1941), 591–592. (Russian)
11. S. J. Leese, *Multifunctions of Souslin type*, Bull. Austral. Math. Soc. 11 (1974), 395–411.

PROFSOJUZNAJA 97-1-203, MOSCOW B-279, U.S.S.R.