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UPPER AND LOWER ESTIMATES ON THE RATE OF CONVERGENCE OF APPROXIMATIONS IN H_p

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Let $1 < p \leqslant \infty$ and let $H_p(U)$ denote the family of all functions f that are analytic in the unit disc U and such that

$$\|f\|_p = \lim_{r \to 1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p} < \infty. \tag{1}$$

Let σ_n be defined by

$$\sigma_n = \inf_{w_j \in C, x_j \in U} \ \sup_{f \in H_p(U), \|f\|_p = 1} \left| \int_{-1}^1 f(x)\, dx - \sum_{j=1}^n w_j f(x_j) \right| . \tag{2}$$

We announce the following result.

THEOREM 1. *Given any $\epsilon > 0$, there exists an integer $n(\epsilon) \geqslant 0$ such that whenever $n > n(\epsilon)$, then*

$$\exp[-(5^{1/2}\pi + \epsilon)n^{1/2}] \leqslant \sigma_n \leqslant \exp\left[-\left(\frac{\pi}{(2q)^{1/2}} - \epsilon\right) n^{1/2}\right], \tag{3}$$

where $q = p/(p - 1)$.

Next, let $H_p^*(U)$ denote the family of all functions g such that $f \in H_p(U)$, where $f(z) = g(z)/(1 - z^2)$, and such that $H_p^*(U)$ is normed by $\|g\|_p^* = \|f\|_p$, where $\|f\|_p$ is defined as in (1). Let $g \in H_p^*(U)$, and let $\{T_n(g)\}$ be a linear approximation scheme defined by

$$T_n(g)(z) = \sum_{j=1}^n g(x_j)\phi_{n,j}(z), \ xj \in U \tag{4}$$

where $\phi_{n,j}$ is analytic in U for each n and j, and such that

$$\|T_n(g)\|_p^* \leqslant C\|g\|_p^* \tag{5}$$

where C is independent of n. We then announce

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THEOREM 2. *Given any $\epsilon > 0$, there exists an integer $n(\epsilon) \geqslant 0$, such that whenever $n > n(\epsilon)$, then*

$$\exp[-(5^{1/2}\pi + \epsilon)n^{1/2}] \leqslant \inf_{T_n g \in H_p^*(U), \|g\|_p^* = 1} \sup_{-1<x<1} |g(x) - T_n(g)(x)| \leqslant \exp\left[-\left(\frac{\pi}{2q^{1/2}} - \epsilon\right)n^{1/2}\right]. \tag{6}$$

Let us briefly mention some papers which are relevant to the present work. In 1964 Wilf [9] proved for the case $p = 2$ that $\sigma_n = O((\log n/n)^{1/2})$. In 1971 Haber [2] and Johnson and Riess [3] proved for $p = 2$ that $\sigma_n = O(n^{-1/2})$. The authors of [2], [3] conjectured that their bound was the best bound possible. In 1973 [6] it was shown by the author that for $p = 2$, $\sigma_n = O(e^{-\pi n^{1/2}}/2)$. In 1975 it was shown by Loeb and Werner [4] that for arbitrary $p > 1$, $\sigma_n \leqslant 2^{1+2/q}\exp[-(n/2)^{1/2}/(2q)]$.

The bounds of Theorem 1 are sharper than any others that have been obtained previously. While there is a gap in the constants of the upper and lower bounds, no one has previously obtained a lower bound. Moreover, no one has previously obtained upper or lower bounds of the type in Theorem 2, for approximation in $H_p^*(U)$.

The results of Theorems 1 and 2 may be extended to establishing the optimal $O(e^{-cn^{1/2}})$ rate of convergence of quadrature and interpolation in other H_p spaces, $p > 1$. In what follows, we shall describe some of these. We shall also mention known methods of quadrature or interpolation in each case, which converge at the $O(e^{-an^{1/2}})$ rate. At this time it is not known whether or not $a = c$ for any of these methods.

(a) Let $0 < d \leqslant \pi/2$, let $\mathcal{D}_d = \{z = x + iy\colon |\arg[(1+z)/(1-z)]| < d\}$. (Note that $\mathcal{D}_{\pi/2} = U$) and let $H_p(\mathcal{D}_d)$ denote the family of all functions f that are analytic in $\mathcal{D}_d$ such that

$$\|f\|_p = \lim_{C \to \partial \mathcal{D}_d, C \subset \mathcal{D}_d} \inf \left(\int_C |f(z)|^p |dz|\right)^{1/p} < \infty. \tag{7}$$

The optimal rate of convergence of quadratures (2) in $H_p(\mathcal{D}_d)$ is $O(e^{-cn^{1/2}})$, where

$$(\pi d/q)^{1/2} \leqslant c \leqslant 5^{1/2}\pi + \epsilon, \qquad \epsilon > 0 \text{ arbitrary}. \tag{8}$$

The quadrature methods of Theorem 1.6(b) of [8] and Theorem 3.2 of [7] converge at the $O(\exp[-(\pi d/q)^{1/2} n^{1/2}])$ rate.

(b) Let $H_p^*(\mathcal{D}_d)$ denote the family of all functions g such that $f \in H_p(\mathcal{D}_d)$

where $f(z) = g(z)/(1 - z^2)$ and where $H_p(\mathcal{D}_d)$ is defined in (a) above. The optimal rate of convergence of interpolation (4) in $H_p^*(\mathcal{D}_d)$ is $O(e^{-cn^{1/2}})$, where

$$(9) \qquad [\pi d/(2q)]^{1/2} - \epsilon \leqslant c \leqslant 5^{1/2}\pi + \epsilon, \qquad \epsilon > 0 \text{ arbitrary.}$$

The method [8]

$$(10) \qquad \begin{cases} g(x) \cong \sum_{j=-N}^{N} g(x_j) S(j, h) \circ \log\left(\frac{1+x}{1-x}\right), \\ S(j, h)(x) = \frac{\sin[\pi(x - jh)/h]}{[\pi(x - jh)/h]}, \\ h = (\pi dq/N)^{1/2}, \ x_j = \tanh(jh/2), \end{cases}$$

converges at the $O(\exp\{-([\pi d/(2q)]^{1/2} - \epsilon)n^{1/2}\})$ rate, where $n = 2N + 1$, and $\epsilon > 0$ is arbitrary.

(c) Let $\mathcal{D}_d = \{z = x + iy\colon |y| < d\}$, and let $H_p(\mathcal{D}_d)$ denote the family of all functions f that are analytic in $\mathcal{D}_d$ such that

$$N(f, y) = \left(\int_R \{|f(x + iy)|^p + |f(x - iy)|^p\} \cosh^{2p/q}(x/2)\, dx\right)^{1/p} < \infty$$

$y < d$, and such that $\|f\|_p = N(f, d^-) < \infty$.

(i) The optimal rate of convergence of n-point quadratures

$$\int_R f(x)\, dx \cong \sum_{j=1}^{n} w_j f(x_j)$$

in $H_p(\mathcal{D}_d)$ is $O(e^{-cn^{1/2}})$, where c is subject to (8). The trapezoidal rule,

$$\int_R f(x)\, dx \cong h \sum_{j=-N}^{N} f(jh), \qquad h = (2\pi dq/N)^{1/2},$$

converges at the $\exp[-(\pi d/q)^{1/2} n^{1/2}]$ rate [8], where $n = 2N + 1$.

(ii) The optimal rate of interpolation of $f \in H_p(\mathcal{D}_d)$ on R is $O(e^{-cn^{1/2}})$, where c is subject to (9). Interpolation via the Whittaker cardinal function,

$$f(x) \cong \sum_{j=-N}^{N} f(jh) S(j, h)(x) \qquad (h = (\pi dq/N)^{1/2})$$

converges at the $O(\exp\{-([\pi d/2q)]^{1/2} - \epsilon)n^{1/2}\})$ rate [8], where $n = 2N + 1$, and $\epsilon > 0$ is arbitrary.

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