BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 1, January 1978

EXISTENCE AND APPLICATIONS OF REMOTE POINTS

BY ERIC K. VAN DOUWEN

Communicated by P. T. Church, May 23, 1977

All spaces are completely regular, X^* *is* $\beta X - X$.

A point p of X^* will be called a *remote point of* X if $p \notin \mathrm{Cl}_{\beta X} D$ for every nowhere dense $D \subseteq X$. Fine and Gillman, [*FG*] showed that **Q**, the rationals, and **R**, the reals, have remote points if *CH* holds; their proof shows that X has remote points if X is separable and not pseudo-compact. We prove the existence of remote points without assuming additional set theoretic axioms, under slightly stronger conditions on X.

Recall that a *π-base* (or *pseudo-base*) for a space X is a family $\mathcal{B}$ of nonempty open sets such that every nonempty open set of X includes a member of $\mathcal{B}$. The *π-weight* of a space is the smallest cardinality for a π-base.

THEOREM A. *If* X *is a nonpseudocompact space with countable π-weight, then* X *has* $2^{\mathfrak{c}}$ *remote points.*

I originally proved this only for $X = \mathbf{Q}$, improving a technique from [$\mathbf{v}D_1$]. I am indebted to Mary Ellen Rudin for showing me how to make my ideas work for $X = \mathbf{R}$. The above theorem is a further improvement.

For applications we need a "pointed" version of extremal disconnectedness.

DEFINITION. *If* $p \in X$, then X is called *extremally disconnected at* p if for all disjoint open $U, V \subseteq X$, $p \notin \overline{U} \cap \overline{V}$.

One can show that βX is extremally disconnected at every remote point of X. Without much effort one deduces the following theorem. (X is *nowhere locally compact* if no point has a compact neighborhood.)

THEOREM B. *Let* X *be a nonpseudocompact space with countable π-weight.*

(a) βX *is extremally disconnected at some point of* X^*.

(b) *If* X *is nowhere locally compact,* X^* *is extremally disconnected at some point.*

Frolík, [*F*], proved *that* X^* is not homogeneous if X is not pseudocompact. Theorem B can be used to show *why* X^* is not homogeneous, for suitable X.

AMS (MOS) subject classifications (1970). Primary 54D35, 54D40; Secondary 54B10, 54G05.

THEOREM C. *Let X be a nowhere locally compact nonpseudocompact separable first countable space. Then X^** is *not homogeneous because X^* is extremally disconnected at some but not at all points.*

This applies if e.g. $X = \mathbf{Q}$, or $X = \{\text{irrationals}\}$, or $X = \{\text{Sorgenfrey line}\}$. As another application we show that certain spaces cannot be factored as a product of spaces without isolated points. The key observation is that $X \times Y$ is not extremally disconnected at any point if X and Y are separable spaces without isolated points.

THEOREM D. *Y is not the product of two spaces without isolated points in each of the following two cases*:

(a) *$Y = \beta X$ for some nonpseudocompact X with countable π-weight*;

(b) *$Y = X^*$ for some nowhere locally compact nonpseudocompact X with countable π-weight.*

(Actually one does not need the condition on the π-weight in (a), see [vD_2], but I do not know if it can be avoided in (b).)

COROLLARY. *$\mathbf{Q}^*$ and $(\mathbf{Q}^*)^\kappa$ are not homeomorphic, for $\kappa \geqslant 2$.*

I do not know if $(\mathbf{Q}^*)^2$ and $(\mathbf{Q}^*)^3$ are nonhomeomorphic.

As yet another application we mention the following curiosities.

EXAMPLE. *There is an extremally disconnected space which has a connected compactification.*

Indeed, if X is any connected nowhere locally compact separable metrizable space, like $\mathbf{R}^\omega$, then the subspace E of all points at which the connected space βX is extremally disconnected turns out to be dense in βX, but then E is extremally disconnected.

For the other application, recall that a space is called *ω-bounded* if every countable subset has compact closure.

THEOREM E. *$\mathbf{R}^*$ is the union of three pairwise disjoint dense ω-bounded subspaces.*

If one calls a point p of X^* a *far point of X* if $p \notin \mathrm{Cl}_{\beta X} D$ for every closed discrete subset D of X, [vD_1], then the three subspaces are the remote points of $\mathbf{R}$, the far points of $\mathbf{R}$ which are not remote and the points of $\mathbf{R}^*$ which are not even far. Under CH there is a family of 2^c such subspaces, [*W*].

ADDED IN PROOF: If κ and λ are cardinals with $\kappa > \lambda \geqslant 1$, then $(Q^*)^\kappa$ and $(Q^*)^\lambda$ are not homeomorphic. The nontrivial proof will appear elsewhere.

REFERENCES

[vD$_1$] E. K. van Douwen, *Why certain Čech-Stone remainders are not homogeneous*, Colloq. Math. (to appear).

[vD$_2$] ———, *When $\Pi\beta$ and $\beta\Pi$ are homeomorphic* (to appear).

[vD$_3$] ———, *Remote points* (in preparation).

[F] Z. Frolik, *Non-homogenity of $\beta P - P$*, Comment. Math. Univ. Carolinae **8** (1967), 705–709.

[FG] N. J. Fine and L. Gillman, *Remote points in βR*, Proc. Amer. Math. Soc. **13** (1962), 29–36. MR **26** #732.

[W] R. G. Woods, *Some $\aleph_0$-bounded subsets of Stone-Čech compactifications*, Israel J. Math. **9** (1971), 250–256. MR **43** #3997.

INSTITUTE FOR MEDICINE AND MATHEMATICS, MATHEMATICS BUILDING, OHIO UNIVERSITY, ATHENS, OHIO 45701