EXISTENCE AND APPLICATIONS OF REMOTE POINTS

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All spaces are completely regular, $X^*$ is $\beta X - X$.

A point $p$ of $X^*$ will be called a remote point of $X$ if $p \notin \text{Cl}_{\beta X}D$ for every nowhere dense $D \subseteq X$. Fine and Gillman, \cite{FG} showed that $\mathbb{Q}$, the rationals, and $\mathbb{R}$, the reals, have remote points if CH holds; their proof shows that $X$ has remote points if $X$ is separable and not pseudo-compact. We prove the existence of remote points without assuming additional set theoretic axioms, under slightly stronger conditions on $X$.

Recall that a $\pi$-base (or pseudo-base) for a space $X$ is a family $B$ of nonempty open sets such that every nonempty open set of $X$ includes a member of $B$. The $\pi$-weight of a space is the smallest cardinality for a $\pi$-base.

**Theorem A.** If $X$ is a nonpseudocompact space with countable $\pi$-weight, then $X$ has $2^\omega$ remote points.

I originally proved this only for $X = \mathbb{Q}$, improving a technique from $[vD]$. I am indebted to Mary Ellen Rudin for showing me how to make my ideas work for $X = \mathbb{R}$. The above theorem is a further improvement.

For applications we need a “pointed” version of extremal disconnectedness.

**Definition.** If $p \in X$, then $X$ is called extremally disconnected at $p$ if for all disjoint open $U, V \subseteq X$, $p \notin \overline{U} \cap \overline{V}$.

One can show that $\beta X$ is extremally disconnected at every remote point of $X$. Without much effort one deduces the following theorem. ($X$ is nowhere locally compact if no point has a compact neighborhood.)

**Theorem B.** Let $X$ be a nonpseudocompact space with countable $\pi$-weight.

(a) $\beta X$ is extremally disconnected at some point of $X^*$.

(b) If $X$ is nowhere locally compact, $X^*$ is extremally disconnected at some point.

FrolíkJ, \cite{F}, proved that $X^*$ is not homogeneous if $X$ is not pseudocompact. Theorem B can be used to show why $X^*$ is not homogeneous, for suitable $X$.

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THEOREM C. Let \( X \) be a nowhere locally compact nonpseudocompact separable first countable space. Then \( X^* \) is not homogeneous because \( X^* \) is extremally disconnected at some but not at all points.

This applies if e.g. \( X = \mathbb{Q} \), or \( X = \{ \text{irrationals} \} \), or \( X = \{ \text{Sorgenfrey line} \} \). As another application we show that certain spaces cannot be factored as a product of spaces without isolated points. The key observation is that \( X \times Y \) is not extremally disconnected at any point if \( X \) and \( Y \) are separable spaces without isolated points.

THEOREM D. \( Y \) is not the product of two spaces without isolated points in each of the following two cases:

(a) \( Y = \beta X \) for some nonpseudocompact \( X \) with countable \( \pi \)-weight;
(b) \( Y = X^* \) for some nowhere locally compact nonpseudocompact \( X \) with countable \( \pi \)-weight.

(Actually one does not need the condition on the \( \pi \)-weight in (a), see [vD\(_2\)], but I do not know if it can be avoided in (b).)

COROLLARY. \( Q^* \) and \((Q^*)^\kappa\) are not homeomorphic, for \( \kappa \geq 2 \).

I do not know if \((Q^*)^2\) and \((Q^*)^3\) are nonhomeomorphic.

As yet another application we mention the following curiosities.

EXAMPLE. There is an extremally disconnected space which has a connected compactification.

Indeed, if \( X \) is any connected nowhere locally compact separable metrizable space, like \( R^\omega \), then the subspace \( E \) of all points at which the connected space \( \beta X \) is extremally disconnected turns out to be dense in \( \beta X \), but then \( E \) is extremally disconnected.

For the other application, recall that a space is called \( \omega \)-bounded if every countable subset has compact closure.

THEOREM E. \( R^* \) is the union of three pairwise disjoint dense \( \omega \)-bounded subspaces.

If one calls a point \( p \) of \( X^* \) a far point of \( X \) if \( p \notin \text{Cl}_{\beta X} D \) for every closed discrete subset \( D \) of \( X \), [vD\(_1\)], then the three subspaces are the remote points of \( R \), the far points of \( R \) which are not remote and the points of \( R^* \) which are not even far. Under \( CH \) there is a family of \( 2^c \) such subspaces, [W].

ADDED IN PROOF: If \( \kappa \) and \( \lambda \) are cardinals with \( \kappa > \lambda > 1 \), then \((Q^*)^\kappa\) and \((Q^*)^\lambda\) are not homeomorphic. The nontrivial proof will appear elsewhere.

REFERENCES

[vD\(_2\)] ———, When \( \Pi \beta \) and \( \beta \Pi \) are homeomorphic (to appear).
[vD\(_3\)] ———, Remote points (in preparation).


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