Let $m \geq 2$ and $r$ be integers, let $y_0$ be an integer in the least residue system mod $m$, and let $\lambda$ be an integer coprime to $m$ with $\lambda \neq \pm 1 \pmod{m}$ and $(\lambda - 1)y_0 + r \neq 0 \pmod{m}$. A sequence $y_0, y_1, \ldots$ of integers in the least residue system mod $m$ is generated by the recursion $y_{n+1} \equiv \lambda y_n + r \pmod{m}$ for $n = 0, 1, \ldots$. In the homogeneous case $r \equiv 0 \pmod{m}$, one chooses $y_0$ to be coprime to $m$. The sequence $x_0, x_1, \ldots$ in the interval $[0, 1)$, defined by $x_n = y_n/m$ for $n = 0, 1, \ldots$, is a sequence of linear congruential pseudo-random numbers. The sequence is purely periodic; let $\tau$ denote its least period. In practice, $m$ is taken to be a large prime or a large power of 2.

For a given $s \geq 2$, the serial test is set up to determine the amount of statistical dependence among $s$ successive terms in the sequence $x_0, x_1, \ldots$. To this end, one considers the $s$-tuples $x_n = (x_n, x_{n+1}, \ldots, x_{n+s-1})$, $n = 0, 1, \ldots$, and measures the deviation between the empirical distribution of the first $N$ of these $s$-tuples and the uniform distribution on $[0, 1]^s$ by the quantity $D_N$ introduced in [3], where $1 \leq N \leq \tau$. For the homogeneous case, effective estimates for $D_\tau$ were established in [3], [4]. By extending techniques from [2] and [4], we can now handle the general case. Estimates for $D_N$ with $N < \tau$ are of great practical interest because in calculations involving linear congruential pseudo-random numbers one only uses an initial segment of the period and not the full period itself.

The number $R^{(s)}(\lambda, m, q)$ is defined as in [3]. $C_s$ will denote an explicitly known constant depending only on $s$, whose exact value may be different in each occurrence.

**Theorem 1.** For a prime $m$ we have

\[
D_N < \begin{cases} 
\frac{s}{m} + \frac{C_s}{\tau} (m - \tau)^{3s}(\log m)^s + \frac{1}{2}R^{(s)}(\lambda, m, m) & \text{for } N = \tau, \\
\frac{s}{m} + \frac{C_s}{N} m^{3s} (\log m)^s + \frac{1}{2}R^{(s)}(\lambda, m, m) & \text{for } 1 \leq N \leq \tau.
\end{cases}
\]
Now let $m$ be a prime power, say $m = p^\alpha$ with $p$ prime and $\alpha \geq 2$. For $h \geq 1$, let $\mu(p^h)$ be the exponent to which $\lambda$ belongs mod $p^h$. Define a positive integer $\beta$ as follows: if $p$ is odd, let $\beta$ be the largest integer such that $p^\beta$ divides $\lambda^{\mu(p)} - 1$; if $p = 2$, let $\beta$ be the largest integer such that $2^\beta$ divides $\lambda^{\mu(4)} - 1$. Furthermore, let $\kappa$ be the largest integer such that $p^\kappa$ divides $\lambda - 1$, let $\omega$ be the largest integer such that $p^\omega$ divides $(\lambda - 1)\gamma_0 + r$, and set $\gamma = \beta + \omega - \kappa$.

**Theorem 2.** For a prime power $m = p^\alpha$, $p$ prime, $\alpha \geq 2$, and a $\lambda$ with $\gamma < \alpha$ we have

$$D_N < \begin{cases} \frac{s}{m} + \frac{1}{2}R^{(s)}(\lambda, m, p^{\alpha-\gamma}) & \text{for } N = \tau, \\ \frac{s}{m} + \frac{C_\alpha}{N} \left( \frac{n\tau}{\mu(m)} \right)^{\gamma_1} (\log m)^{s+1} + \frac{1}{2}R^{(s)}(\lambda, m, p^{\alpha-\gamma}) & \text{for } 1 \leq N < \tau. \end{cases}$$

We note that in the frequently used special case $m = 2^\alpha$, $\lambda \equiv 5 \pmod 8$, and $r$ odd we have $\gamma = 0$. The interpretation of these results is similar to that in [3], [4].

The quantity $\rho^{(s)}(\lambda, m)$ introduced in [3] is convenient for computational purposes. Because of the above results and [3, Theorem 4], the reciprocal of $\rho^{(s)}(\lambda, m)$ may be taken as a measure for the amount of statistical dependence among $s$ successive terms in a sequence $x_0, x_1, \ldots$ having a large period $\tau$. The fact that this is really the correct indicator is shown by the following result.

**Theorem 3.** For any $m$, $\lambda$, and $N$ with $1 \leq N \leq \tau$ we have

$$D_N \geq \begin{cases} 1/\pi \rho^{(s)}(\lambda, m) & \text{for } 2 \leq s \leq 6, \\ \pi/2(2\pi + 1)^s \rho^{(s)}(\lambda, m) & \text{for } s \geq 7. \end{cases}$$

We remark that the estimates for $D_N$ given here yield effective error bounds for Monte Carlo integrations using the points $x_0, x_1, \ldots, x_{N-1}$ as nodes. This follows from general inequalities for the integration error in terms of $D_N$ which can be found in [1, Chapter 2, §5].

**REFERENCES**