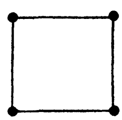
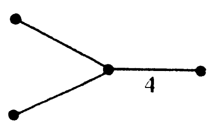
BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 3, May 1978

Three-dimensional nets and polyhedra, by A. F. Wells, Wiley, New York, 1977, xii + 268 pp., $29.95.

In a storage room of the University of Minnesota, there used to be (and probably still are) four large isosceles triangular mirrors, with edges proportional to $2 : \sqrt{3} : \sqrt{3}$, relics of an abandoned film project. If they were put together as faces of an 'isosceles' tetrahedron (with some device to prevent the sloping mirrors from sagging under their own weight), and if you could look in through a hole in one of the edges, you would see a remarkable array of images. For this *tetragonal disphenoid* is one of the three kinds of tetrahedron that can serve as a fundamental region for a reflection group [Coxeter 1973, p. 84; Shubnikov and Koptsik 1974, p. 201]. It (or the group) is denoted by a 'Dynkin symbol'



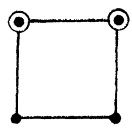
in which the four dots represent the four mirrors while the four links indicate dihedral angles $\pi/3$ between pairs of mirrors. The mirrors represented by 'opposite' dots are at right angles because those dots are not directly linked. A plane that bisects one of these two right angles dissects the disphenoid into two congruent pieces, each of which is a *trirectangular* tetrahedron



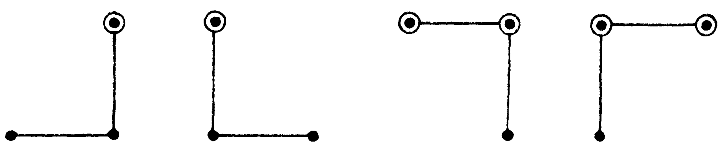
The link marked 4 indicates that the cutting plane, which we naturally replace by a mirror, forms a dihedral angle $\pi/4$ with its neighbor. (Instead of a link marked 4, Witt [1941, p. 301] prefers a double link.) This smaller tetrahedron still has a plane of symmetry, bisecting one of its right angles, and this can be used to dissect the tetrahedron into two enantiomorphous pieces, each of which is an *orthoscheme*

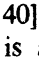
This is the remaining kind of 4-mirror kaleidoscope. It is marked *CMIO* in the reviewer's Fig. 2.2A [Coxeter 1974, p. 13]. It combines with its image in the vertical plane *MIO* to form the trirectangular tetrahedron *CBIO*. This, in turn, combines with its image in the horizontal plane *CBI* to form the disphenoid *CBOO′*, which is where this discussion began.

By inserting rings round one or more of the dots, we symbolize a 'uniform honeycomb' whose vertices are all the images of a point that lies on all the 'unringed' mirrors, in a position equidistant from all the 'ringed' mirrors [Coxeter 1940, pp. 390–404]. In particular, if only one dot is ringed, the chosen point is one vertex of the tetrahedron; and if all four dots are ringed, it is the incenter of the tetrahedron. This notation has the advantage of showing at a glance the nature of the various cells of the honeycomb: we simply suppress each dot in turn and see what uniform polyhedron (or polygon) is symbolized by the rest of the graph [Coxeter 1974, pp. 16–18, 164]. This happens because any three of the four mirrors form a *Fedorov kaleidoscope* [Shubnikov and Koptsik 1974, pp. 68–69]. For instance, there is a honeycomb, denoted by $q\delta_4$, whose graphical symbol



indicates that its typical vertex is the midpoint of one of the four short edges of the disphenoid (say, the midpoint of CO). Suppressing each dot in turn, we obtain the 'subsymbols'



The first two represent tetrahedra; the last two, truncated tetrahedra. The truncated tetrahedron [Ball and Coxeter 1974, p. 140] is bounded by four triangles  and four hexagons . Thus $q\delta_4$ is a honeycomb whose cells consist of these two solids, occurring with equal frequency.

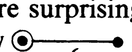
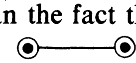
Symbols such as $q\delta_4$ seem rather artificial till one realizes that they remain valid in higher spaces [Coxeter 1968 p. 46]: δ_{n+1} is the n-dimensional cubic lattice whose vertices have as vertices all combinations of n integers; $h\delta_{n+1}$ has half these vertices, and $q\delta_{n+1}$ (q for 'quarter') has half the vertices of $h\delta_{n+1}$. These symbols, with suitable subscripts after the h or q, are a convenient adaptation of the 'contraction' and 'expansion' symbols invented by Mrs. Stott [see Coxeter 1978], who was the middle one of George Boole's five daughters. She wrote $c(\frac{1}{2}e_0)$ for h, $c(\frac{1}{2}e_0)(\frac{1}{2}e_3)$ for q, $c(\frac{1}{2}e_0)e_2e_3$ for $h_{2,3}$, and so on.

The occurrences of these uniform honeycombs in Wells's book, and in other related works, are summarized in the accompanying Table on p. 469. The graphical symbols in the first column are followed by the abbreviated symbols (such as δ_4) and the 'point symbols' (such as 4^{12}) which indicate the plane faces (interfaces between pairs of solid cells) that occur at each vertex. The fourth column refers to Andreini [1905], the fifth to Wells, and the remaining two to Wachman, Burt and Kaufmann [1974], who, following Wells's example, extracted from the set of plane faces (of all except $h_3\delta_4$) a polyhedral surface in which every edge belongs to just two faces. As a trivial instance, we may extract from the cubic honeycomb $4^{12} = \delta_4$ all the squares that lie in one plane; they form the 'squared paper' tessellation $4^4 = \delta_3$. What happens at each vertex may be described in terms of the *vertex figure* of the honeycomb. The vertex figure of δ_4 is an octahedron whose twelve edges lie in the planes of the twelve squares that surround the typical vertex. The extracted polyhedral surface has for its vertex figure a polygon (usually a skew polygon) whose edges occur among those of the vertex figure of the honeycomb: in the present case, an 'equatorial' square of the octahedron.

More interestingly, John Flinders Petrie (when he was nineteen) extracted from the same cubic honeycomb 4^{12} an infinite skew polyhedron or 'regular sponge' $4^6 = \{4, 6 \mid 4\}$, whose vertex figure is a skew hexagon: a 'Petrie polygon' of the octahedron [Coxeter 1939, p. 242; 1968, pp. 76–77; 1973, p. 32]. Petrie noticed that this regular sponge 4^6 (with square 'holes' or 'tunnels') has a dual $6^4 = \{6, 4 \mid 4\}$. This 6^4 is extracted in an analogous manner from the honeycomb $4^2 6^4 = t_{1,2}\delta_4$, whose vertex figure is a tetragonal disphenoid having edges of various lengths representing the 2 squares and 4 hexagons that meet at one vertex of $4^2 6^4$. This disphenoid yields a skew quadrangle when any two opposite edges are removed. If these are the edges representing the two squares, the skew quadrangle is regular and the sponge is the above-mentioned 6^4. If instead the removed edges represent two of the four hexagons, the skew quadrangle is kite-shaped, and the sponge (no longer 'regular' but only 'uniform') has the point symbol $(4.6)^2$, indicating that the cycle of four faces round each vertex consists of a square, a hexagon, another square, and another hexagon; in other words, each hexagon is surrounded by six squares, and each square by four hexagons. [Compare Ball and Coxeter 1974, p. 136, where the cuboctahedron is denoted by $(3.4)^2$.]

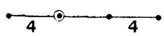
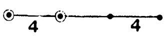
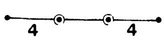
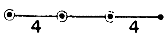
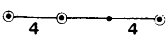
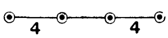
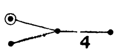
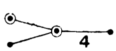
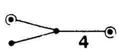
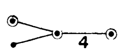
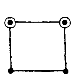
A third regular sponge, $6^6 = \{6, 6 \mid 3\}$, can be extracted from the honeycomb $3^6 6^6 = q\delta_4$, whose vertex figure is a tall triangular antiprism. The lateral edges of this antiprism form a regular skew hexagon which is the vertex figure of the sponge 6^6. (The remaining edges of the antiprism yield the triangular holes which are indicated by the 3 in the symbol $\{6, 6 \mid 3\}$.)

The honeycomb $q\delta_4$ may be described as having integral Cartesian coordinates whose residues modulo 4 are (0, 0, 0) or (0, 1, 1) or (0, 2, 2) or (0, 3, 3) or (1, 2, 3), in any order [Coxeter 1968, p. 47]. If each of the tetrahedral cells is dissected into four congruent triangular pyramids (with their common apex at its centre), and four such pyramids are stuck onto the triangular faces of each truncated tetrahedron, we obtain a honeycomb of congruent cells each having 16 faces. Michael Goldberg has conjectured that this 16-hedron, unhappily named 'snub-tetrahedron' by Keith Critchlow [1969, p. 54], has the greatest possible number of faces for a convex space-filler.

The attentive reader, looking at the Table, may wonder why the 'square' Dynkin symbol does not appear with one or three or four rings, or with two *opposite* dots ringed. The answer is that the honeycombs so indicated merely repeat those already listed: one ring yields $h\delta_4$, two opposite rings $t_1\delta_4$, three rings $h_2\delta_4$, and four rings $t_{1,2}\delta_4$ [Coxeter 1940, pp. 402, 403]. Such duplications are no more surprising than the fact that the regular hexagon is equally well denoted by  or 

The Table continues with further infinite uniform polyhedra whose faces do not belong to any uniform honeycomb. This list is certainly not complete. For instance, Branko Grübaum has extracted from δ_4 a new 4^5, different from all Wachman's eleven. Also J. R. Gott [1967, p. 498] found four new 'pseudopolyhedrons' 3^8, 3^{10}, 4^5, 5^5, the last of which makes a particularly interesting model.

TABLE OF UNIFORM HONEYCOMBS AND INFINITE POLYHEDRA

Honeycombs		Andreini	Wells	Wachman et al.	Polyhedra
 $\delta_4 = t_0\delta_4$	4^{12}		Fig. 17.21A Fig. 17.15	67 87	$4^6 = \{4, 6\|4\}$ 4^5
 $t_1\delta_4$	$3^8 4^4$	Fig. 18	Fig. 10.1(b)	73, 81 82 92	$3^3 4.3.4$ $(3.4)^4$ $(3^3 4)^2$
 $t_{0,1}\delta_4$	$3^4 8^4$	Fig. 17	Fig. 10.1(a)	93	$3^3 8^2$
 $t_{0,2}\delta_4$	$3^2 4^7$	Fig. 20	p.146	70 99	$(3.4^2)^2$ 3.4^5
 $t_{1,2}\delta_4$	$4^2 6^4$	Fig. 14	Fig. 17.21B	68 77	$6^4 = \{6, 4\|4\}$ $(4.6)^2$
 $t_{0,1,2}\delta_4$	$4^3 6^2 8$	Fig. 21	Fig. 18.4	69 101	$4^2 6^2$ $4^2 6.8$
 $t_{0,1,3}\delta_4$	$3.4^5 8^2$	Fig. 22	Fig. 10.2(a)	71 96	3.4^4 $4^3 8^2$
 $t_{0,1,2,3}\delta_4$	$4^3 6.8^2$	Fig. 24^{bis}	Fig. 18.5	66 72 97	$4.8.6.8$ $4^3 6$ $(4.8)^2$
 $h\delta_4$	3^{24}	Fig. 12	'F lattice' Fig. 17.4	25 83	3^9 3^9
 $h_2\delta_4$	$3^2 4^2 6^2 8^2$	Fig. 24	Fig. 10.2(b)	80	$3.4.6^2 4$
 $h_3\delta_4$	$3^3 4^6$	Fig. 19	p. 146	71	3.4^4
 $h_{2,3}\delta_4$	$3.4.6^2 8^2$	Fig. 23	Fig. 12.2	100	$6^2 8^2$
 $q\delta_4$	$3^6 6^6$	Fig. 15	Fig. 17.21C	76	$6^6 = \{6, 6\|3\}$
			Fig. 17.9 Fig. 17.13 Fig. 17.12 Fig. 17.7 Fig. 17.10 Fig. 17.22 Fig. 17.19 Fig. 17.18 Fig. 17.20	90 91 74 84 85 45 20 46 88	3^{12} 3^9 3^8 3^8 3^7 4^6 4^5 4^5 4^5

Wells generalizes the notion of an infinite polyhedron to a 'net' in which the only polygons that occur are *skew* polygons. The most famous case (see p. 117) is the 'diamond net' [Hilbert and Cohn-Vossen, 1952, p. 49]; its vertices have integral coordinates (x, y, z) where $x \equiv y \equiv z \pmod 2$ and $x + y + z \equiv 0$ or $1 \pmod 4$. In other words, Wells is looking for *graphs*, of given valency and girth, which can be realized in Euclidean space so as to be symmetrical by translations in three independent directions. The number of possibilities is so great that complete enumeration seems to be out of the question. The search is justified by his observation that such graphs provide structural formulae for more than fifty crystals (which he lists on p. 265). He illustrates many of them by pairs of stereoscopic photographs, and some by very accurate drawings; see especially his pp. 147, 170, 254 and 255.

BIBLIOGRAPHY

A. Andreini 1905, *Sulle reti di poliedri regolari e semi-regolari*, Mem. Soc. Ital. Sec. (3) **14**, 75–129.

W. W. R. Ball and H. S. M. Coxeter 1974, *Mathematical recreations and essays*, 12th ed., Univ. of Toronto Press, Toronto, Ont.

H. S. M. Coxeter 1939, *The regular sponges, or skew polyhedra*, Scripta Math. **6**, 240–244.

______ 1940, *Regular and semiregular polytopes*. I, Math. Z. **46**, 380–407.

______ 1968, *Twelve geometric essays*, Southern Illinois Univ. Press, Carbondale, Ill.

______ 1973, *Regular polytopes*, 3rd ed., Dover, New York.

______ 1974, *Regular complex polytopes*, Cambridge Univ. Press, New York.

______ 1978, *Polytopes in the Netherlands*, Nieuw Arch. Wisk. (3) **26** (to appear).

Keith Critchlow 1969, *Order in space*, Thames and Hudson, London; Viking Press, New York.

J. R. Gott, III, 1967, *Pseudopolyhedrons*, Amer. Math. Monthly **74**, 497–504.

D. Hilbert and S. Cohn-Vossen 1952, *Geometry and the imagination*, Chelsea, New York.

A. V. Shubnikov and V. A. Koptsik 1974, *Symmetry in science and art*, Plenum, New York.

A. Wachman, M. Burt and M. Kleinmann 1974, *Infinite polyhedra*, Technion, Haifa.

Ernst Witt 1941, *Spiegelungsgruppen und Aufzählung halbeinfacher Liescher Ringe*, Abh. Math. Sem. Univ. Hamburg **14**, 289–322.

H. S. M. COXETER

BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 3, March 1978

Discrete multivariate analysis: Theory and practice, by Yvonne M. M. Bishop, Stephen E. Fienberg, and Paul W. Holland with the collaboration of Richard J. Light and Frederick Mosteller, MIT Press, Cambridge, Mass., 1975. Second printing with corrections, 1976, x + 557 pp., $30.00.

Categorical data arise whenever counts, rather than continuous measurements, are made. Such data are especially important in the social sciences, in which qualitative responses to surveys are frequently a source of information, and in medicine, in which classification of patients, treatments, and/or symptoms and judgements with respect to outcomes are the variables of interest. Analysis of categorical data has a long history, beginning at least with Karl Pearson's famous paper (1900). Contributions of major significance were made by R. A. Fisher (e.g., 1934, 1936, 1941), Bartlett (1935), and Birch (1963). From its inception, the analysis of categorical data has, more than many areas of statistics, emphasized multivariate aspects, although many