British mathematical books have wretched indices. This one maintains the tradition.

## REFERENCES

1. R. C. Buck, *Expansion theorems for analytic functions*. I, Lectures on Functions of a Complex Variable (W. Kaplan, M. O. Reade and G. S. Young, Editors), Univ. of Michigan Press, Ann Arbor, 1955, pp. 409–419; p. 410.

2. D. G. Bourgin, *A class of sequences of functions*, Trans. Amer. Math. Soc. **60** (1946), 478–518.

3. S. V. Bočkarev, *Existence of a basis in the space of functions analytic in the disk, and some properties of Franklin's system*, Mat. Sb. **95** (137) (1974), 3–18, 159.

4. L. Carroll, *Through the looking-glass*, Chapter VI (In The Complete Works of Lewis Carroll, Nonesuch Press and Random House, London and New York, n.d.).

5. P. Enflo, *A counterexample to the approximation problem in Banach spaces*, Acta Math. **130** (1973), 309–317.

6. W. S. Gilbert, *The Mikado*, Act 2 (In The Savoy Operas, Macmillan, London, 1926, p. 371).

7. J. T. Marti, *Introduction to the theory of bases*, Springer-Verlag, New York, 1969.

8. I. Singer, *Bases in Banach spaces*, Springer, Berlin, 1970.

R. P. BOAS

BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 4, July 1978


*Brownian motion, Hardy spaces and bounded mean oscillation*, by K. E. Petersen, London Mathematical Society Lecture Note Series (2) vol. 28, Cambridge University Press, Cambridge, London, New York, Melbourne, 105 pp.

In recent years the techniques and theorems of Brownian motion have been used to prove theorems about harmonic and analytic functions. It is always pleasant when two branches of mathematics which ostensibly have little to do with one another can help each other out. There are two main links which allow Brownian motion (roughly representing the paths of an idealized random traveller) to be connected to the theory of harmonic and analytic functions. Kakutani [4] showed that Brownian motion can be used to solve the Dirichlet problem. Dispensing with the technicalities of continuity, smoothness, and measurability, here is what Kakutani's theorem says: Let $S$ be an open set in $\mathbf{R}^n$ and let $u$ be a real-valued function defined on $\partial S$. Let $z \in S$ and consider a typical Brownian path $\gamma_z$ starting at $z$. Let $s(\gamma_z)$ denote the point of $\partial S$ at which $\gamma_z$ first hits $\partial S$. Define $\hat{u}(z)$ to be the average value of $u(s(\gamma_z))$, where the average is taken over all Brownian paths $\gamma_z$. Then $\hat{u}$ is a harmonic function on $S$ with boundary values $u$.

A theorem of Lévy [5] links Brownian motion to analytic functions defined in the plane. This thoerem states that a nonconstant analytic function composed with Brownian motion is also Brownian motion, although the time scale must be changed on each Brownian path. The inituition behind Lévy's result is that an analytic function preserves angles, so that the randomness of direction is preserved. Since an analytic function need not preserve lengths, an adjustment of the time scale is necessary.

For $0 < p < \infty$ and $u$ a function defined on the open unit disk $D$ of the complex plane, define

$$\|u\|_p = \sup_{r<1} \left[ \int_0^{2\pi} |u(re^{i\theta})|^p \, d\theta \right]^{1/p}.$$

The Hardy space $H^p$ is the set of analytic functions $f$ satisfying $\|f\|_p < \infty$. The set of real-valued harmonic functions $u$ satisfying $\|u\|_p < \infty$ is denoted by $h^p$. If $u$ is a real-valued harmonic function, then the harmonic conjugate of $u$ is the unique real-valued function $\tilde{u}$ such that $u + i\tilde{u}$ is analytic and $\tilde{u}(0) = 0$.

The major part of the book under review is devoted to answering the following question: Given a harmonic function $u$, how can one determine, without computing $\tilde{u}$, whether $u + i\tilde{u} \in H^p$? For $1 < p < \infty$ the answer has been known for a long time; M. Riesz [6] showed ($1 < p < \infty$) that $u + i\tilde{u} \in H^p$ if and only if $u \in h^p$.

Fix a number $\sigma \in (0, 1)$. For each $e^{i\theta} \in \partial D$, let $\Omega_\sigma(\theta)$ denote the interior of the convex hull of $e^{i\theta}$ and the disk of radius $\sigma$ centered at the origin. The nontangential maximal function $N_\sigma u$ is defined by $N_\sigma u(e^{i\theta}) = \sup\{|u(z)|: z \in \Omega_\sigma(\theta)\}$.

Hardy and Littlewood [3] showed that if $0 < p < \infty$ and $u + i\tilde{u} \in H^p$, then $N_\sigma u \in L^p(\partial D, d\theta)$. Most of Petersen's book is devoted to proving Burkholder, Gundy and Silberstein's [1] converse to this theorem; so for $0 < p < \infty$, $u + i\tilde{u} \in H^p$ if and only if $N_\sigma u \in L^p(\partial D, d\theta)$.

The statement of the above theorem contains no mention of Brownian motion; however, the proof depends heavily on it. For each Brownian path $\gamma$ beginning at the origin let $u^*(\gamma) = \sup\{|u(\gamma(t))|: t \in [0, T_\gamma]\}$, where $T_\gamma$ is the first time $t$ that $\gamma(t)$ hits $\partial D$. A connection between the nontangential maximal function $N_\sigma u$ and the Brownian maximal function $u^*$ is made by proving that $N_\sigma u \in L^p(\partial D, d\theta)$ if and only if $u^* \in L^p(B)$, where $B$ is the set of all Brownian paths with the usual Wiener measure. The harmonic conjugation operator is brought in by proving that $u^* \in L^p(B)$ if and only if $(\tilde{u})^* \in L^p(B)$. Putting these two theorems together shows that

$$N_\sigma u \in L^p(\partial D, d\theta) \Rightarrow u \in L^p(B) \Rightarrow (\tilde{u})^* \in L^p(B) \Rightarrow N_\sigma \tilde{u} \in L^p(\partial D, d\theta).$$

Since $N_\sigma \tilde{u} \in L^p(\partial D, d\theta)$ trivially implies that $\tilde{u} \in h^p$, the Burkholder, Gundy, Silverstein result ($N_\sigma u \in L^p(\partial D, d\theta) \Rightarrow \tilde{u} \in h^p$) follows immediately.

In the final chapter Petersen discusses $H^p$ martingales, gives a Brownian motion characterization of BMO (the space of functions of bounded mean oscillation), and gives a probabilistic proof of Fefferman and Stein's [2] theorem that the dual of $H^1$ is BMO.

It is difficult to write a book cutting across two fields and thus appealing to two different audiences. Petersen should be commended for making this important and interesting material accessible to both potential audiences. For probabilitists who might not have a background in function theory, he has included a chapter on Hardy spaces. For analysts who might not have a background in probability, he has included a chapter on Brownian motion. Both of these background chapters contain only statements of results without proofs. A reader approaching this material for the first time might want to supplement one of these chapters (depending on the reader's field) with more detailed sources.

REFERENCES

1. D. L. Burkholder, R. F. Gundy and M. L. Silverstein, *A maximal function characterization of the class $H^p$*, Trans. Amer. Math. Soc. **157** (1971), 137–153.

2. C. Fefferman and E. M. Stein, *$H^p$ spaces of several variables*, Acta Math. **129** (1972), 137–193.

3. G. H. Hardy and J. E. Littlewood, *A maximal theorem with function-theoretic applications*, Acta Math. **54** (1930), 81–116.

4. Shizuo Kakutani, *Two-dimensional Brownian motion and harmonic functions*, Proc. Imp. Acad. Tokyo **20** (1944), 706–714.

5. Paul Lévy, *Processus stochastiques et mouvement Brownien*, Gauthier-Villars, Paris, 1948.

6. M. Riesz, *Über Pontezreiben mit vorgeschriebenen Anfangsgliedern*, Acta Math. **42** (1920), 145–171.

SHELDON AXLER

BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 4, July 1978


*Orderable groups*, by Roberta Botto Mura and Akbar Rhemtulla, Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, Inc., New York and Basel, 1977, iv + 169 pp., $19.75.

The study of ordered groups began at the end of the last century. One of the first important results was obtained by Hölder in 1901 in a paper that investigated the measurement of physical data. He used the cuts introduced by Dedekind to show that an archimedean ordered group is isomorphic to the additive group $R$ of the real numbers. Thus the real number system is the maximal archimedean ordered group. In 1907 Hahn proved that an ordered abelian group can be embedded into a lexicographic product of copies of $R$. His proof necessarily starts from scratch, is about 40 pages long and is one of the more difficult proofs in mathematics. In the 50's and 60's several new proofs were derived and the theorem was extended to partially ordered abelian groups and even to partially ordered sets.

Hahn realized that his lexicographic products were ordered fields provided that the index set is an ordered group. In fact, most of the early papers on ordered groups are related to the theory of ordered fields. This led to the beautiful Artin-Schreier theory of real closed fields (1926) and to the solution of Hilbert's 17th problem. Later Mal'cev (1948) recognized the connection between ordered groups and the embedding of integral domains into division rings and cancellative semigroups into groups, Neumann and others constructed ordered division rings by extending Hahn's ideas to nonabelian groups. In particular Mal'cev (1948) and Neumann (1949) showed that the group ring of an ordered group over an ordered division ring can be embedded in an ordered division ring, Hilbert in his Grundlagen der Geometrie showed that each ordered group can be embedded in an ordered division ring.

In the 30's and 40's the theory of ordered abelian groups branched out into two areas: (I) partially ordered abelian groups and rings, and (II) ordered nonabelian groups, which is the subject of these notes. In 1935 Kantorovich started his investigation of partially ordered linear spaces, which was continued through the war years by Kantorovich and his pupils. Also during