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Orderable groups, by Roberta Botto Mura and Akbar Rhemtulla, Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, Inc., New York and Basel, 1977, iv + 169 pp., $19.75.

The study of ordered groups began at the end of the last century. One of the first important results was obtained by Hölder in 1901 in a paper that investigated the measurement of physical data. He used the cuts introduced by Dedekind to show that an archimedean ordered group is isomorphic to the additive group R of the real numbers. Thus the real number system is the maximal archimedean ordered group. In 1907 Hahn proved that an ordered abelian group can be embedded into a lexicographic product of copies of R. His proof necessarily starts from scratch, is about 40 pages long and is one of the more difficult proofs in mathematics. In the 50's and 60's several new proofs were derived and the theorem was extended to partially ordered abelian groups and even to partially ordered sets.

Hahn realized that his lexicographic products were ordered fields provided that the index set is an ordered group. In fact, most of the early papers on ordered groups are related to the theory of ordered fields. This led to the beautiful Artin-Schreier theory of real closed fields (1926) and to the solution of Hilbert's 17th problem. Later Mal′cev (1948) recognized the connection between ordered groups and the embedding of integral domains into division rings and cancellative semigroups into groups, Neumann and others constructed ordered division rings by extending Hahn's ideas to nonabelian groups. In particular Mal′cev (1948) and Neumann (1949) showed that the group ring of an ordered group over an ordered division ring can be embedded in an ordered division ring, Hilbert in his Grundlagen der Geometrie showed that each ordered group can be embedded in an ordered division ring.

In the 30's and 40's the theory of ordered abelian groups branched out into two areas: (I) partially ordered abelian groups and rings, and (II) ordered nonabelian groups, which is the subject of these notes. In 1935 Kantorovich started his investigation of partially ordered linear spaces, which was continued through the war years by Kantorovich and his pupils. Also during the war years Iwasawa, Nakano, Birkhoff and Lorenzen wrote fundamental papers on partially ordered groups and in particular on lattice-ordered groups. The concept of lattice-ordered groups goes back to Dedekind's study of divisibility relations, and is now fundamental in certain branches of functional analysis. The theory of topological vector spaces, Banach lattices and the ring $C(X)$ of continuous functions on a topological space X all make use of the natural partial order that is present.

In the early 30's Krull replaced the real numbers by ordered groups and ordered division rings in the theory of valuations. In the late 40's Birkhoff, Iwasawa and Neumann each showed that a free group admits an order, and in 1949 Vinogradov proved that a free product of ordered groups admits an order which extends the order of the original groups. This, of course, led to the question: which classes of groups can be ordered? Among them are the classes of torsion free abelian groups, torsion free nilpotent groups, free solvable groups and free polynilpotent groups. Reiger (1946) and Mal′cev (1949) showed that the chain of convex subgroups is fundamental in the study of ordered groups.

An interesting subclass of orderable groups is that of O^*-groups. These are groups in which each partial order can be extended to an order. This class is large enough to include the torsion free nilpotent groups and solvable ordered groups with finite rank. A chapter is devoted to the study of these groups.

A group is an R^*-group if $g^{x_1} \cdots g^{x_n} = e$ implies $g = e$ for all g, $x_1, \ldots, x_n$ in G. Each ordered group is an R^*-group and it was long conjectured that the converse is true. V. V. Bludov showed in 1974 that this is false and a different counterexample due to the authors is included in these notes.

During the 60's and to the present the Russian school of group theorists have been very active and successful in the study of ordered groups, especially Kokorin and Kopytov. A fair amount of these notes is devoted to their work. Also included is Phillip Hall's theorem that each ordered group can be embedded in an algebraically simple ordered group, and Neumann's theorem that a countable ordered group can be embedded in a two generator ordered group. Hall's proof consists of a very clever application of the wreath product.

In summary these notes are well written and many of the proofs are much more elegant than those in the literature. It is only assumed that the reader has had a basic graduate course in group theory, for these notes are only concerned with "the interplay between orderability properties and group theoretical conditions, such as nilpotency, solvability and finiteness of rank". Moreover, "this approach is narrow and does not include many important aspects, such as topological properties, Hahn's embedding theorem and lattice-ordered groups". The only competition is from the book by Kokorin and Kopytov: *Fully ordered groups* (Wiley and Sons, 1974). This is a translation of the original 1972 Russian edition, which includes Hahn's theorem and its corollaries and a study of the open interval topology of an ordered group. These notes however contain material that appeared after 1970. A combination of the two books gives a complete up to date survey of the theory of ordered groups, and contains enough material for a one semester course or seminar.

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Theory of optimal search, by Lawrence D. Stone, Mathematics in Science and Engineering, vol. 118, Academic Press, New York, 1975, xiv + 260 pp., $29.50.

That resources are limited and must be carefully allocated among competing ends, each in itself desirable, is a central fact of the world we live in. The analysis of the resource allocation problem for society as a whole has been a central concern of economic theory, while its study from more limited and more detailed perspectives has become perhaps the major focus of the field termed operations research or management science. Biological study, particularly the field of ecology and some aspects of evolutionary theory, has also put some emphasis on the resource allocation problems of living creatures; after all, Charles Darwin ascribed his notion of natural selection to the influence of the economist, Thomas R. Malthus, whose emphasis on the implications of resource limitations earned for economics the name of "the dismal science."

The book under review is a study of optimal resource allocation in a particular field, search for an object when the search process uses up scarce resources. This particular theory arose during World War II as the problem of locating enemy submarines and was studied by a group headed by the probability theorist, B. O. Koopman. Most of the subsequent interest has also been motivated by seeking submarines, including lost friendly ones. There is a considerable literature, to which the author has been a major contributor, and now we have a survey which is indeed admirable in scope and exposition.

Although the author concentrates on his particular area of resource allocation, more general problems are implied, and some of the theorems are widely applicable. Nevertheless, the search problem in its elementary forms has special features which enable stronger results to be obtained than are available generally.

The resource allocation problem with a single scarce resource can be stated as follows: Let there be a finite or denumerable set, J, of possible activities. Each can be operated at alternative levels indexed by a real number (possibly restricted to the integers, if the activity can be carried on only in discrete steps, or to some other subset of the reals). Let f be a mapping from J to the range of activity levels. If z is the activity level for the jth activity, let $c(j, z)$ be the amount of the scarce resource used in the jth activity. Hence, if f is the specification of activity levels, the total amount of the resource used is,

$$\sum_{j \in J} c(j, f(j)).$$

If the total amount of the resource is considered to be limited then we are