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Introduction to ergodic theory, by Ya. G. Sinai, Princeton Univ. Press, Princeton, New Jersey, 1977, 144 pp., $6.00.

The author has endeavored to present the general results of ergodic theory by examining special cases. His very considerable success testifies to the care and insight with which his examples, illustrating the methods and basic concepts of ergodic theory, have been chosen. The examples are, moreover, explained very clearly and at a level which should make the book accessible to a wide audience. The reader should be warned, however, that some of the results appear on first reading to be simpler than they really are, and that not all areas of ergodic theory are treated. The last section of this review will discuss a particularly important omission.

Ergodic theory arose from efforts to abstract some mathematically interesting aspects of dynamical systems. Two such systems, which are very closely connected, may be studied as examples. Consider first an ideal gas whose molecules are subject to the laws of classical mechanics and which are enclosed in a container. Statistical mechanics consists of the study of this system, and especially of the limiting behavior of its properties as the number of molecules tends to infinity. As a second example, consider a planetary system also subject to the laws of classical mechanics. Celestial mechanics deals with the study of such planetary systems. The second example differs from the first merely in that the case of interest is not the limiting one, and in that there are no collisions against the walls of a container. Ergodic theory is, to a large extent, the study of ideas which have their origin in statistical or celestial mechanics.

We proceed now to the concept of phase space, which has come to be a crucial idea in the study of dynamical systems. Phase space does not correspond to the physical space of the dynamical system. It is rather a representation space, each point of which corresponds to a state of the system. The position and velocity of each of the N molecules completely describes the state of the system consisting of a gas composed of N molecules. Since three coordinates are required to describe the position and three coordinates to describe the velocity of each of the N molecules, phase space of the system consists of $R_3^N \times R_3^N = R_{6N}$, where R_k is a k-dimensional Euclidean space. For a gas enclosed in the container C_3, phase space can be further specified to be $C_3^N \times R_3^N$.

The same phase space can be used for the planetary system of N planets, C_3 denoting a volume of space large enough always to contain the system. In both cases, the system traces out a trajectory in the phase space, starting from the point which represents the initial conditions of the system and passing through the collection of points which represent the conditions of the system as they develop with time. If one imagines all possible initial conditions of a system and the trajectories emanating from the points which represent them, one has the flow in phase space induced by the system, each point being carried by the trajectory defined by the point and the given dynamical system.

A classical result, Liouville's theorem, says that flows of this type have the property that they preserve volume in phase space. Since the total energy of the system must be preserved, trajectories are contained in those hyper-surfaces whose energy is equal to the initial energy of the system. It follows from Liouville's theorem that flows also preserve surface hyper-area when the flow is restricted to an energy hyper-surface, and it can easily be seen that energy hyper-surfaces are finite for the ideal gas. This observation permits the consideration of finite measure spaces.

The use of phase space in the study of dynamical systems was introduced by the work of Poisson (1809), Lagrange (1810), Cauchy (1819), Hamilton (1834) and Liouville (1838). Hamilton discovered the principle and equations which are named after him, and his work may be regarded as the definitive treatment of phase space in the study of dynamical systems. His equations constitute the basis for much of the present treatment of dynamics and simplify many results. Liouville's theorem, for example, can be proved very simply from Hamilton's equations.

The properties of the set of trajectories corresponding to a given dynamical system may be viewed from three different perspectives, the abstraction of which has given rise to the three main branches of ergodic theory. These three branches are differential dynamics, topological dynamics and measure dynamics–distinguished, of course, according to whether the perspective is that of differential equations, of topology, or of measure theory. Differential and topological dynamics originated with the work of Poincaré and Birkhoff. Poincaré in his *Mémoire* of 1881 for the first time used topological ideas to formulate and solve problems in mechanics.

Differential dynamics studies the geometric properties of the trajectories of differential equations, and topological dynamics, going further in abstraction, studies the properties of topological transformation groups. Both emphasize methods and results which hold in the special case resulting from dynamical systems. It was Birkhoff who in the 1920's extended the work of Poincaré and began the systematic study of topological dynamics. There is a story I cannot resist telling which also links the work of Poincaré with that of Birkhoff. In 1858 Dirichlet told Kronecker of having discovered an approximation method for treating problems in mechanics and of having proved the stability of planetary systems. Weierstrass, Kovalevski and Mittag-Leffler unsuccessfully attempted to reconstruct Dirichlet's results, as he died without publishing his work. Mittag-Leffler then persuaded the King of Norway and Sweden to establish a prize for the solution of the problem, which was formulated as finding a series solution valid for all time for the N-body problem–the problem of finding the solution for the planetary system of N planets. In 1889 the prize was awarded to Poincaré for his essay on celestial mechanics, although he too had failed to solve the problem. The N-body problem remains open, although Sundman in 1908 solved it for the case $N = 3$. Poincaré's topological ideas were developed in several papers, the last of which appeared in 1912. In that paper he conjectured a very interesting geometric theorem and showed that the existence of periodic orbits for the restricted three-body problem would follow from it. Shortly after Poincaré's death, Birkhoff proved Poincaré's conjecture.

The Poincaré-Birkhoff theorem may be stated very simply. If T is a one-to-one continuous area-preserving transformation of the region contained between two concentric circles C_1 and C_2 which maps C_1 onto C_1 and C_2 onto C_2 and in opposite directions, then T has a fixed point. The proof is more delicate than one would suspect.

Statistical mechanics originated principally with the work of Boltzmann, summarized in [2]. This work and that of Gibbs led to the von Neumann and to the Birkhoff ergodic theorems on replacing time means with phase means for ergodic flows. The results stem from Liouville's theorem, and they, together with related results, form measure dynamics. This branch of ergodic theory deals with a finite measure space (later extended to a σ-finite measure space) and a measure preserving transformation $T: X \to X$. The transformation T induces a positive contraction P on all the $L_p(X, \mathfrak{F}, \mu)$ spaces simultaneously, defined by setting $Pg(x) = g(T^{-1}x)$. In 1932 von Neumann proved the L_2-convergence of the averages $A_n(g) = (1/n)\Sigma_{k=0}^{n-1}P^k g$ to an invariant L_2 function for g in L_2, and in 1931 Birkhoff proved the almost everywhere convergence of the averages $A_n(g)$ to an invariant L_1 function for g in L_1. It may be seen that in the ergodic case, when the invariant sets have measure either zero or one, the Birkhoff ergodic theorem implies that time means may be replaced by phase means, for almost all x.

The Poincaré recurrence theorem was originally proved earlier and independently of these results, although it follows from the Birkhoff theorem. The Poincaré theorem says that in the ergodic case, each set of positive measure is visited infinitely often for almost every x under the action of $\{T^k\}$.

Sinai's book begins by discussing the problem of the existence of an invariant measure, and several results are obtained which yield such a measure. The first section contains a proof of the Bogoliubov-Krilov theorem which yields the existence of an invariant measure for a continuous transformation of a compact topological space into itself. There is a discussion of the existence of a smooth invariant measure for diffeomorphisms of smooth compact manifolds, and a later section of the book contains a sketch of the proof of Liouville's theorem mentioned earlier, based on Hamilton's equations.

Sinai also considers translations of the torus, giving a proof of the Weyl-von Neumann theorem on the ergodicity of such translations, and a second proof, without using Fourier series, of the ergodicity of an irrational rotation of the circle. There is also a discussion of the ergodic properties of the one-dimensional model of an ideal gas, of geodesic flows on Riemannian manifolds, of the billiard ball problem, and of measure-preserving transformations arising in the theory of probability.

Together with his proof of Poincaré's conjecture of 1912, Birkhoff showed that many Lagrangian dynamical systems are isomorphic to a billiard ball moving on a flat billiard table with a boundary curve determined by the system. We may define a table T to be any compact convex body in the plane bounded by a continuously differentiable curve ∂T. The billiard ball, idealized as a point, moves in a straight line with unit speed until it hits the boundary δT, where it rebounds making an angle of reflection equal to the angle of incidence. Birkhoff observed that the motion of the billiard ball defines a flow on the interior of T and that the application of the Poincaré-Birkhoff theorem to yield periodic orbits is particularly transparent in this case. Professor Sinai's section on the billiard ball problem deals with these matters.

The extent to which any mathematical theory gives a clearer understanding of physical and mathematical reality constitutes an important consideration in evaluating its merit. As J. T. Schwartz observed in this context [4], "the intellectual attractiveness of the mathematical argument as well as the considerable mental labor involved in following it makes mathematics a powerful tool of intellectual prestidigitation, a glittering deception in which some are entrapped and some, alas, entrappers." It must be admitted that, thus far, general ergodic theory has not been applied to physics with great success. For example, in spite of the fact that the ergodic theorem, as originally formulated and proved by Birkhoff, was motivated by statistical mechanics, not a single result of real physical interest has been shown to follow from it. This sort of difficulty appears to be the rule rather than the exception.

It must also be said, however, that there are deep and significant applications of ergodic theory to other branches of mathematics. Sinai's book reflects this achievement, though not, perhaps, with sufficient force, as the examples overall have a physical orientation. Among the most significant applications of ergodic theory to other branches of mathematics, apart from topology, have been the results of Kolmogorov and Sinai himself, connecting entropy to the isomorphism problem, and the results on the potential theory of a Markov process. The notion of entropy has deepened the understanding of measure spaces and the types of transformations which they admit. The last three chapters of Sinai's book are concerned with entropy questions, where the entropy of a dynamical system is defined and then calculated for the dynamical system given by a billiard ball on a polygonal table and for the flow over a two-dimensional torus.

In order to explain the applications to Markov processes it is necessary to extend the theory to trajectories in the space of distributions (in the sense of probability theory) induced by the semigroup of a Markov process–for example, Brownian motion. For measure dynamics thus extended, the objects of interest become positive contractions of L_1 instead of measure-preserving transformations of the underlying space. The proof of the maximal ergodic theorem may then be viewed as a statement about the potential theory, the balayage and the stopping times of the underlying Markov process. There are many other results as well which unite measure dynamics, potential theory and the theory of Markov processes. Regrettably, Sinai's book includes no discussion of these ideas, though it would have been quite possible to develop them in keeping with the spirit of the book, by considering special cases–the random walk, for example.

Professor Sinai's lectures are beautifully written. Our criticism may be summarized by saying simply that they end too soon. We hope that Professor Sinai will publish a sequel adding problems which will illustrate more clearly the mathematical applications of ergodic theory and which will go further in developing the theory in general terms.

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Vorlesungen über numerische Mathematik, by Heinz Rutishauser, Birkhäuser Verlag, Basel, Switzerland, Bands 1 and 2, 1976, 164 pp. and 229 pp., Fr/DM 40,48.

The two volumes under review here are elementary lecture notes on numerical analysis written by Heinz Rutishauser before his premature death in 1970 at the age of fifty-two. Although Rutishauser intended ultimately to publish these notes as a textbook, they were by no means in final form at his death, and in spite of the able editorship of Martin Gutknecht they remain somewhat rough-hewn and not a little out of date. Nonetheless, Rutishauser was one of the most successful and respected workers in this field, and it is not surprising that his notes represent one of the best introductions to numerical analysis as it is actually practiced.