It has been said that the supreme occurrence in the course of an idea is that brief moment between the time it is heresy and the time it is trite. At the start of the 1960s “nonlinear functional analysis” seemed to strike most mathematicians as a contradiction in terms but by the end of that decade, some functional analysts were apologizing for considering “only the linear case.”

During this period many began to realize (or to rediscover from earlier times) that quite a number of pressing scientific problems are nonlinear in nature. At the same time many others began to realize (or again, to rediscover) that many nonlinear problems have a vigorous algebraic life.

It is widely understood that many linear problems have a natural setting in some ring of linear transformations. For two illustrations (out of a vast number of possibilities) consider the following:

1. A study of the spectrum of a bounded selfadjoint operator $T$ on a Hilbert space $H$ leads naturally to a consideration of the smallest closed subring of $L(H, H)$ which contains $T$.

2. A strongly continuous one-parameter semigroup of bounded linear transformations on a Banach space $X$ may be considered as a kind of ray in the ring $L(X, X)$. By now a substantial start for both nonlinear spectral...
theory and for one-parameter semigroups of nonlinear transformations has
been made but a setting for such developments (parallel to that provided by
rings of linear operators for linear problems) has not been clearly established.

To look for such a setting it is natural to turn to the subject of near-rings:
A near-ring is a set $N$ with an addition and a multiplication so that $N$ is a
group (not necessarily abelian) under addition and is a semigroup under
multiplication so that right distributivity holds, i.e.,

$$(x + y)z = xz + yz, \quad x, y, z \in N.$$  

An example of a near-ring is the set of continuous real-valued functions on
all of $R$ with addition defined pointwise and multiplication taken to be
composition. Clearly left distributivity does not hold here so that one
certainly does not have a ring.

In a sense, near-rings provide a nonlinear analogue to the developments of
linear algebra. Certainly linear algebra is a subject of vast applicability but it
also has great importance as a source of ideas for linear functional analysis.
One is reminded that linear functional analysis grew out of linear algebra in
response to the needs of linear differential equations. It is likely that know­
ledge of near-rings together with a good assessment of the needs of modern
nonlinear differential equations will lead to fruitful developments in
nonlinear functional analysis such as a “nonlinear Gelfand theory” or a much
more extensive nonlinear spectral theory. Near-rings of Lipschitz, zero-fixing
transformations from a Banach space to itself seem to seem to be potentially valuable
for applications.

In passing from linear to nonlinear problems the matter of notation seems
particularly important. Linear analysis has long possessed a good notation.
For a linear transformation $T$ on a vector space no one speaks of the “linear
transformation $T(x)$” where $x$ is supposed to be a “variable” vector. Contrast
this with the curse of “the function $f(*)$” which is still being peddled by
almost all calculus books. The wide-spread persistence of this deficient
functional notation seems a hindrance to general recognition of the
underlying function-algebraic aspects of many problems.

This book contains a nearly encyclopedic account of the present state of
the algebraic theory of near-rings. It is nearly self-contained for one familiar
with ring theory.

To gain some initial insight into near-rings consider the relevant notion of
ideal. For a near-ring $N$, a normal subgroup $I$ of $(N, + )$ is an ideal in $N$ if
($\alpha$) $IN \subseteq I$ and ($\beta$) $n(n' + I) - n'n \in I, n, n' \in N$. If ($\alpha$) is satisfied then $I$
is a right ideal and if ($\beta$) is satisfied then $I$ is a left ideal. An ideal turns out
to be a homomorphic image of $N$. The lack of symmetry between left and
right ideals is typical for the subject.

Closely connected with near-rings are $N$-groups. For a near-ring $N$ and
group $(\Gamma, + )$, $\mu: N \times \Gamma \to \Gamma$ gives an $N$-group $(\mu N \equiv \mu(n, \gamma), \gamma \in \Gamma, n \in N)$
provided that

$$(n + n')\gamma = n \gamma + n' \gamma, \quad (nn')\gamma = n(n'\gamma), \quad n, n' \in N, \gamma \in \Gamma.$$  

Such an $N$-group is denoted by $N\Gamma$.

Various decomposition theorems for near-rings and $N$-groups are given
under assumptions of various chain conditions. As in the whole subject, ring
theory gives always at least a rough idea of what to expect.

We describe here some of the ideas which indicate some structure theory
for near-rings. A better description undoubtedly could be done by a card-
carrying algebraist but here it is anyway.

An \( N \)-group \( _N \Gamma \) is simple if and only if it has no nontrivial ideals and it is
called \( N \)-simple if and only if it has no \( N \)-subgroups except \( \Omega \) and \( \Gamma \)
(\( \Omega \equiv \{n0\}, n \in N \)). Simple \( N \)-groups \( _N \Gamma \) such that \( N \Gamma \neq \{0\} \) do not in
general have the property that \( N\gamma = \{0\} \) or \( N\gamma = \Gamma \) for all \( \gamma \in \Gamma \) (unlike the
ring-module case). An \( N \)-group \( _N \Gamma \) is monogenic if there is \( \gamma \in \Gamma \) such that
\( N\gamma = \Gamma \). Three classes of monogenic \( N \)-groups are given—all of which coincide
with irreducibility in the case of ring-modules. Type 0, \( _N \Gamma \) is simple;
Type 1, \( _N \Gamma \) is simple and strongly monogenic (for all \( \gamma \in \Gamma, N\gamma = \{0\} \) or \( \Gamma \));
Type 2, \( _N \Gamma \) is \( N_0 \)-simple (where \( \Gamma \) is considered as an \( N_0 \)-group, \( N_0 \) zero
symmetric, i.e., \( n0 = 0 \) for all \( n \in N_0 \)). For an \( N \)-group \( _N \Gamma \) and subsets \( \Delta_1, \Delta_2 \) of \( N \), define the quotient \( (\Delta_1 : \Delta_2)_N = \{n \in N | n\Delta_2 \subseteq \Delta_1 \} \). A theorem
of S. D. Scott is given: Suppose \( N \) is zero-symmetric, has the DCCN and \( M \)
is a subgroup of \( N \) such that \( NM \subseteq M \). If \( M \) is monogenic (by \( m_0 \)) then \( M \) has
a right identity and \( (0, m_0)M = \{0\} \). Another theorem of Scott is the following
for zero-symmetric near-rings with DCCN: If \( I \) is a minimal ideal, then \( I \) is a
finite direct sum of \( N \)-isomorphic minimal left ideals of \( N \).

For a given near-ring one considers faithful and simple \( N \)-groups based
upon \( N \). For \( \gamma = 0, 1, 2 \), \( N \) is called \( \gamma \)-primitive on \( _N \Gamma \) provided \( _N \Gamma \) is
faithful and of type \( \gamma \). \( N \) is called \( \gamma \)-primitive provided there is \( \Gamma \) so that \( N \) is
\( \gamma \)-primitive on \( _N \Gamma \). An ideal \( I \) in \( N \) is \( \gamma \)-primitive provided \( N/I \) is \( \gamma \)-primitive.
These notions of primitivity are crucial in a description of extensive (but still
incomplete) work toward a density theorem for near-rings.

For a near-ring \( N \) and for \( \gamma \neq 0, 1, 2 \), consider \( \gamma \)-groups \( \Gamma \) of type \( \gamma \).
Consider the intersection of sets \( \{n \in N | n\gamma = 0\} \) over such \( \Gamma \). This inter-
section gives a \( \gamma \)-radical. Its “size” is indicative of how far \( N \) is from being
“\( \gamma \)-semisimple.” Characterizations of such radicals are given along with much
other information concerning radicals in near-rings. The discussion of rad-
cals completes the central part of the book—the part on structure theory.

Various special classes of near-rings are studied. Among these are distribu-
tively generated near-rings, transformation near-rings and near-fields. An
element \( d \in N \) is distributive if \( d(n + n') = dn + dn' \) for all \( n, n' \in N \). A near-ring is called distributively generated if there is a subsemigroup of \( (N_d,+ ) \)
(\( d \) generating \( N, + \) ) where \( N_d \) is the collection of distributive elements of \( N \).
Among other unsettled questions it is asked whether or not every zero-
symmetric near-ring is embeddable into some distributively generated near-
ring.

This book is an algebra book. Its subtitle is “The theory and its
applications.” The applications given are mainly to other parts of algebra and
a few are given to geometry—not exactly what comes to mind to most these
days when “applications” are mentioned. The reader may gather from the
first part of this review that the reviewer feels there will eventually be a host of
real applications—applications real in the sense that linear algebra has real
applications.
In fact, the time seems reasonably near for an historically noteworthy combination of the algebraic theory of near-rings with the fields of nonlinear differential equations, nonlinear functional analysis and numerical analysis. One would be mistaken to dismiss the subject of near-rings as just a haven for out-of-work ring theorists.

J. W. Neuberger


One of the challenges to system theory posed by present day technological, environmental and societal processes is to overcome the increasing rise and complexity of the relevant mathematical models. The amount of computational effort needed to analyze a dynamic process usually increases much faster than the size of the corresponding systems. Consequently the problems arising in large scale systems become either very difficult or uneconomical to solve even with modern computers. In view of this, it has recently been recognized that, for the purposes of stability analysis, control, optimization and so forth, it may be fruitful to decompose a large scale system into a number of interconnected subsystems, to fully utilize the information of these individual subsystems, and to combine this knowledge with interconnection constraints to obtain a solution of the original problem of the large scale system. Thus taking advantage of the special structural features of a given large scale system, one hopes to devise feasible and efficient piece-by-piece methods for solving such systems which are intractable or impractical to tackle by one shot methods.

It is well known that using a single Lyapunov function and the theory of differential inequalities, a variety of problems in the qualitative theory of differential equations can be studied. However, the usefulness of this approach is limited when applied to problems of higher dimension and complex interconnecting structures. It is also known that, in such situations, employing a vector Lyapunov function instead of a scalar Lyapunov function, is more advantageous since it offers a more flexible mechanism and demands less rigid requirements on each component of the vector Lyapunov functions. Many multivariable systems are composed of relatively simple subsystems or aggregates. By grouping variables of a large economy, for example, into a relatively small number of subeconomies, the economy is decomposed into several interconnected subsystems. Stability of the entire economy may be predicted by testing the lower order aggregated comparison subsystems.

Let $E_i$, $i = 1, 2, \ldots, N$, be Banach spaces. Consider the system of differential equations

$$x'_i = f_i(t, x_i), \quad x_i(t_0) = x_{i0}$$

where $f_i \in C[R^+ \times E_i, E_i]$. Let $F_i \in C[R^+ \times E, E_i]$ where $E = E_1 \times E_2$.