This book combined with the book on the same subject by Siljak [**11**] which is also just published and which presents several specific applications of large scale systems to real problems, should form a welcome addition to the subject and help stimulate further work in this new and exciting field.

Bibliography

1. V. M. Matrosov, *On the theory of stability of motion*, Prikl. Mat. Meh. **26** (1962), 992–1002.

2. ______, *The comparison principle with a vector valued function*.I, II, and III, Differential'nye Uravnenija **4** (1968), 1374–1384, 1739–1752; **5** (1969), 1171–1185.

3. ______, *Vector Lyapunov functions in the analysis of nonlinear interconnected systems*, Symp. Math. Roma **6**, Mechanica Nonlineare Stabilita 1970, Academic Press, New York (1971), 209–242.

4. ______, *Comparison method in system dynamics*, Differential'nye Uravnenija **10** (1974), 1547–1559; **11** (1975), 403–417.

5. V. Lakshmikantham, *Several Lyapunov functions*, Proc. of the Fifth Internat. Conference on Nonlinear Oscillation, Kiev, U.S.S.R., 1969, pp. 268–275.

6. ______, *On the method of vector Lyapunov functions*, Proc. Twelfth Annual Allerton Conference on Circuit and System Theory, October, 1974, pp. 71–76.

7. P. Habets and K. Peiffer, *Classification of stability like concepts and their study using vector Lyapunov functions*, J. Math. Anal. Appl. **44** (1973), 537–570.

8. D. D. Siljak, *Stability of large scale systems*, Proc. of the IFAC Fifth World Congress, 1972, pp. 1–11.

9. ______, *Competitive economic systems; stability, decomposition and aggregation*. Proc. of the IEEE Conf. on Decision and Control, 1973, pp. 265–275.

10. ______, *Stability of large scale systems under structural perturbations*, Proc. IEEE Trans. on Systems, Man and Cybernetics **3** (1972), 657–663.

11. ______, *Large scale dynamic systems; stability and structure*, North-Holland, Amsterdam, 1977.

12. D. D. Siljak and M. B. Vukecvie, *Large scale systems: Stability complexity, reliability*, J. Franklin Inst. **301** (1976), 49–69.

13. G. S. Ladde, *Cellular systems*–I, *Stability of chemical systems*–II, *Stability of compartmental systems*, Math. Biosciences **29** (1976), 309–330; **30** (1976), 1–21.

14. ______, *Stability of model ecosystems with time-delay*, J. Theoret. Biol. **61** (1976), 1–13.

15. G. S. Ladde and D. D. Siljak, *Connective stability of large scale stochastic systems*, Internat. J. Systems Sci. **6** (1975), 713–721.

16. ______, *Stability of multispecies co-munity in ramdomly varying environment*, J. Math. Biol. **2** (1975), 165–178.

17. G. S. Ladde and D. D. Siljak, *Stochastic stability and instability of model ecosystems*, Proc. IFAC Sixth World Congress, 1975, pp. 1–7.

18. D. D. Siljak, *On the stability of arms race*, Chapter 9, in Mathematical Systems in International Relations Research, Edited by J. V. Gillespie and D. A. Zimmes, Praeger Publishers, New York, 1977.

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Linear representations of finite groups, by Jean-Pierre Serre, Graduate Texts in Mathematics, vol. 42, Springer-Verlag, New York, Heidelberg, Berlin, 1977, x + 170 pp., $12.80.

As one so often discovers in tracing the history of a beautiful mathematical idea, we have Gauss to thank for pointing the way towards the representation theory of finite groups. In his discussion of class groups of even binary quadratic forms of given discriminant over the integers, he attempted to attach to each class what he considered to be its "character", in order to distinguish among genera. His description was rather cumbersome, so it fell to Dedekind, inspired by Dirichlet's notation for Gauss' characters, to give the more familiar interpretation of characters as numerical functions. The general notion of irreducible characters of abelian groups was set down in 1879, in the third edition of Dirichlet and Dedekind's *Vorlesungen über Zahlentheorie*, and the inclusion of the idea in Weber's famous algebra text made it familiar to many mathematicians of the late nineteenth century.

In 1886, Dedekind introduced the group determinant, which is suggested by the study of discriminants of Galois extensions. Let the finite group G have elements $g_1, \ldots, g_n$, and let $X_{g_i} = X_i$ be an indeterminant, for $i = 1, \ldots, n$. The group determinant is the polynomial $F(X_1, \ldots, X_n) = \det(X_{g_i g_j^{-1}})$, whose coefficients may be regarded as elements of any ring with identity. Dedekind showed that if G is abelian, then

$$F(X_1, \ldots, X_n) = \prod_{i=1}^{n} \left(\sum_{s=1}^{n} \chi_s(g_i) X_i \right),$$

where $\chi_1, \ldots, \chi_n$ are the irreducible characters of G, i.e. the homomorphisms of G into the multiplicative group of nonzero complex numbers. For certain nonabelian groups, he could obtain factorizations of F into linear terms by allowing coefficients in suitably chosen noncommutative algebras.

The matter reappeared about ten years later, when Frobenius, motivated by similar ideas in the theory of theta functions and encouraged by correspondence with Dedekind, undertook further analysis of the group determinant. He obtained a factorization

$$F(X_1, \ldots, X_n) = \prod_{s=1}^{k} F_s^{e(s)},$$

where F_s is an irreducible complex polynomial of degree $f^{(s)}$. He then discussed k, $e(s)$ and $f^{(s)}$ in terms of the structure of G. In the discussion, there appeared the complex-valued functions which would soon be revealed as the irreducible characters of G. In 1897, Frobenius introduced matrix representations of groups. The idea is simple: a representation of degree m of G is a homomorphism from G to the group $GL(m, C)$ of invertible $m \times m$ complex matrices. Such a representation is *irreducible* if no nontrivial subspace of the space of column vectors of length m is invariant under all the matrices in the image of the homomorphism. With each representation $\sigma: G \to GL(m, C)$ is associated a function called the *character* of σ; its value at g is the trace of $\sigma(g)$. A character is said to be irreducible if it arises from an irreducible representation.

Frobenius proved many interesting properties of the characters and of other functions on the group which arise from the representations. At the same time, others began working independently on the theory; some gave other proofs of Frobenius' results, while others discovered new phenomena. For instance, Molien had published a thesis in 1892 in which he obtained a number of results about algebras over the complex field, including the information nowadays deduced from Wedderburn's structure theory for semisimple rings. Molien observed that any group determines the algebra that we now call the complex group algebra CG of G. This algebra has the elements of G as a basis, with multiplication determined by that of the group. His methods showed that this algebra was isomorphic to a direct sum of full matrix algebras, and hence that any representation was a direct sum (in the obvious sense) of irreducible ones. He then studied the irreducible representations by his methods, and obtained many of the basic properties of irreducible characters. Burnside also found proofs of the basic facts, using a rather indirect approach through Lie theory in some cases. Maschke, using E. H. Moore's construction of inner products invariant under a group action, showed that any invariant subspace of a representation has an invariant direct complement.

Once the foundations of the theory had been set, work began in earnest on the applications of it to group theory and to other parts of mathematics. The pioneers worked out many of these applications themselves. Thus, Frobenius studied the special class of finite groups now known by his name, and attacked the problem of estimating the number of elements of given order in a group by use of characters. Burnside showed that a group whose order is divisible by only two distinct primes is solvable, and was led to frame his famous conjecture (settled in the affirmative many years later by Feit and Thompson, in a *tour de force* of character theory) that groups of odd order are solvable. To point out the power of character theory, it should be noted that Burnside's two primes theorem has only recently been given a character-free proof; the main results on Frobenius groups are still inaccessible without representation theory. Many other things were done as well. For instance, Frobenius, Schur and Young gave various separate discussions of the characters of the symmetric groups, while Blichfeldt found many clever results on induced representations and on the eigenvalues of elements of finite linear groups.

In time, more conceptual progress was inevitable. In 1929, Noether firmly established the idea that representations can, and perhaps should, be viewed as modules over group algebras, as was suggested in the neglected work of Molien. Thus, let K be a field, G a finite group, and form the K-algebra KG, as was done above for $K = C$. If $\sigma\colon G \to \mathrm{GL}(V)$ is a representation of G by invertible linear transformations on a K-vector space V (which is clearly equivalent to the idea of a matrix representation as described earlier), then V becomes a KG-module by

$$\left(\sum_{g\in G} k_g g\right)v = \sum_{g\in G} k_g\sigma(g)(v).$$

Conversely, if V is given as a KG-module, multiplication by $g \in G$ is a K-linear mapping on V; denoting this by $\sigma(g)$, we obtain a homomorphism from G into $\mathrm{GL}(V)$. This interpretation permits the use of module-theoretic language, in which many statements become more natural, and their proofs more simple. Also, the scope of inquiry is enlarged, as one may replace the field K by a commutative ring R, and make use of non-R-free RG-modules, which do not correspond to matrix representations, but which occur naturally in applications.

Another important contribution was made in 1937 by Clifford, who presented the basic facts about restrictions of representations to normal subgroups, and their relationship to the theory of induced representations (a topic of central importance, which is perhaps too complicated to describe here in an intuitively understandable way). Clifford's results were later extended and refined in difficult works of Mackey, Dade, Fell, Green and others.

The most remarkable progress in the thirties was the development by Brauer of modular representation theory. If K is a field of finite characteristic p, and p does not divide the order of G, then the representations of G over K behave much like representations in characteristic zero. However, if p does divide the order of G, then KG is not a semisimple ring, and the classical theory no longer applies. For instance, Maschke's theorem fails. Hence, there are modules which are indecomposable, but not irreducible. Indeed, if G has a noncyclic p-Sylow subgroup, then there are infinitely many nonisomorphic indecomposable KG-modules, and their K-dimensions are unbounded. Brauer wanted to use the K-representations (*modular representations*) to study complex representations and group theory. A great deal of care must be taken to gain useful information in the modular case. For instance, the direct sum of p copies of any K-representation has character which is zero everywhere, and hence useless. Brauer circumvented this problem by a very clever construction. If σ is a K-representation and $g \in G$ is p-regular (i.e. the order of g is not divisible by p), then the eigenvalues of $\sigma(g)$ are roots of unity in the algebraic closure of K. These roots of unity are lifted suitably to complex roots of unity, which are then summed. This gives a complex-valued function on the p-regular elements of G, called the *Brauer character* of σ. Brauer and Nakayama showed that these functions have many properties analogous to those of ordinary complex characters.

The most important idea in modular representation theory is that of *blocks*. Although the modular group algebra KG is not semisimple, it is easily seen to be expressible uniquely as a direct product of indecomposable bilateral ideals, called blocks. These blocks give rise to partitions of various sets associated with G: the set of complex irreducible characters, the set of irreducible Brauer characters, the set of indecomposable KG-modules, etc. To illustrate, let $KG = B_1 + \cdots + B_t$ be the decomposition of KG into blocks. The identity element decomposes correspondingly as $e_1 + \cdots + e_t$, where e_i is the identity element of B_i, a primitive idempotent of the center of KG. If M is an irreducible KG-module, then $M = e_iM$ for a unique i; we say M belongs to B_i. It can be shown that if K is taken appropriately, then there is an algebraic number field F and a valuation ring R of F such that all complex representations of G can be realized by matrices over R, and K is isomorphic to R/P, where P is the unique maximal ideal of R. If χ is a complex irreducible character of G, it comes from an RG-module M. One can then show that all KG-composition factors of M/PM belong to the same block B_i, and we say that χ belongs to B_i. This theory has led to many results, most rather technical, on the structure and characterization of simple groups, and has also flourished in its own right.

In recent years, the influence of modern ring and module theory has been evident in research on group representations. Notions from homological algebra and algebraic K-theory have clarified many features of the modular theory, as well as the difficult integral representation theory, which deals with representations over various types of integral domains. Work goes on on all parts of the subject, and there is still a great deal to be discovered.

Serre's book gives a fine introduction to representations for various audiences. It is divided in three parts. The first was originally an appendix to a book on quantum chemistry by Gaston Berthier and Josiane Serre. It gives an exposition of the basics of complex characters and representations, in a style suitable for nonspecialists. There are also a few remarks on the extension of the theory to compact groups.

The second part is for a more sophisticated reader. It gives more detailed information on complex characters, and then proceeds to deeper topics. These come under two main headings. First, there is a discussion of induction theorems, which tell when characters of a group can be obtained in a natural way from characters of certain subgroups. Second, rationally questions in characteristic zero are considered. Thus, one sees what happens when the complex field is replaced by a subfield which may be too small to realize all the complex representations.

The third part is an exposition of Brauer's modular theory. Here, categorical notions (projective covers, Grothendieck groups) are used freely. The connection between complex, integral and modular representations is examined very elegantly, and the Fong-Swan Theorem on lifting modular characters of p-solvable groups is obtained as an application. The Brauer characters are discussed briefly, but block theory is omitted altogether.

Despite the brevity of the book and its omission of many topics, the specialist can profit greatly from reading it. As always with Serre, the exposition is clear and elegant, and the exercises contain a great deal of valuable information that is otherwise hard to find. Also, the discussion of rationality questions is by far the best available. The translation, by L. L. Scott, Jr., is excellent; the design and typography are up to Springer-Verlag's superb standards. Thus, although the book is no substitute for the encyclopedic works of Curtis and Reiner and of Dornhoff, it is highly recommended for specialists and nonspecialists alike.

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Applied and computational complex analysis. II, by Peter Henrici, Wiley, New York, London, Sydney, Toronto, 1977, ix + 662 pp., $32.50.

What would you put into a text for a second course in complex analysis? I expect that most of us, faced with this decision, would follow Hille in accepting some material as canonical and pursue our personal interests for the rest. Hille's basic list consisted of analytic continuation, elliptic functions, entire and meromorphic functions, normal families, and conformal mapping, but was for a rather "pure" course. Suppose it is to be a course oriented toward applications, meaning applications outside of mathematics itself? One has to consider what the applicable parts of the subject are (now, not in some